

# Cubes and cubical chains and cochains in combinatorial topology

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# Preface

**Simplices and cubes.** The present paper can be considered as a continuation of author's paper [I] devoted to the lemmas of Alexander and Sperner and their applications, but is independent from [I]. As a motivation, it is worth to quote the first few phrases of [Sp].

Lebesgue has shown that the invariance of the dimension and also the invariance of domains can be reduced in a simple way to a fundamental theorem of topology (see Section 1). This theorem was also already proved by Brouwer and Lebesgue. But both these proofs present significant difficulties to the readers who are not familiar with this circle of questions. In the present work, first of all, a new, simple proof of this important theorem is given in Sections 1 – 3. It turns out to be appropriate to place the  $n$ -dimensional simplex at the center of consideration instead of the  $n$ -dimensional cube.

In the present paper  $n$ -dimensional cubes are put back at the center of consideration. It begins by a step back from Alexander and Sperner to Lebesgue papers [L1], [L2], one of the main sources of inspiration for Sperner. One could say that the present paper is devoted to cubical analogues of lemmas of Alexander and Sperner, if not the fact that the first such analogues were proved by Lebesgue several years earlier than these lemmas.

**A fragment of history.** The “fundamental theorem of topology”, referred to by Sperner, is the Lebesgue covering theorem or, rather, two covering theorems. See Section 1. Lebesgue results were originally published [L1] as an excerpt from a letter to O. Blumenthal, one of the editors of *Mathematische Annalen*, complementing Brouwer's paper [Br1] about the topological invariance of the dimension. In his letter Lebesgue suggested an approach to the topological invariance based on a completely new idea. Namely, Lebesgue suggested to prove the topological invariance of the dimension by using coverings of domains by small sets and investigating the pattern of intersections of these small sets.

This idea had very far-reaching influence. One can trace to it not only the notion of the covering dimension, but also the notion of the nerve of a covering and, eventually, the notion of a sheaf. In contrast, the influence of Lebesgue proofs was rather limited. It looks like almost nobody read his detailed exposition [L2], in contrast with the outline [L1]. The only exceptions known to the author are W. Hurewicz [H2], who modified Lebesgue proofs, and L. Lusternik and L. Schnirelmann [LS], who applied Lebesgue methods to a different problem. Lusternik and Schnirelmann did not refer to Lebesgue, but the influence of [L2] is quite obvious; they even use some of Lebesgue notations.

Lebesgue main technical tools were the standard partitions of a given cube in  $\mathbb{R}^n$  into small cubes of equal sizes and an approximation of closed sets by unions of these small cubes. The use of cubes by Lebesgue strongly contrasted with the domination of simplicial methods in

topology. But somewhat later it became a desirable feature at least in some quarters. This lead to searching for cubical analogues of Sperner's results and methods. See, for example, the works of A.W. Tucker [T1], [T2], H.W. Kuhn [Ku], and Ky Fan [Fa]. Apparently, none of these mathematicians realized that such analogues existed before Sperner's work.

**Lebesgue, Brouwer, and Sperner.** Brouwer deemed Lebesgue outline [L1] faulty and two years later published [Br2] a fairly cumbersome simplicial version of Lebesgue theorem. See [D], Section II.5 for a modern rendering of Brouwer's version. Brouwer also included in [Br2] an almost page-long footnote devoted to a counterexample to Lebesgue proofs (but not to the theorems). The validity of this counterexample depends on interpreting French *un ensemble* (a set) as German *ein Teilkontinuum* (a connected closed subset). But Lebesgue accepted Brouwer's critique (see [L2], pp. 264–265) and was much more careful in [L2].

Brouwer's proof of his simplicial version is compressed into 13 lines. J. Dieudonne writes that “the proof uses the properties of the degree of a map, and, as usual, is very sketchy and has to be interpreted to make sense” (see [D], Section II.5). In contrast, Lebesgue paper [L2] is self-contained and detailed. If it appears to be demanding, then not because of prerequisites, but precisely because of its elementary language. At the same time Lebesgue proofs are direct and natural. Sperner [Sp] replaced his direct methods by a special case of Brouwer's degree theory admitting an independent proof. See [I] for the details.

**Lebesgue methods and their extensions and applications.** The arguments of Lebesgue and Lusternik–Schnirelmann can be substantially clarified by using the language of cubical chains. This language is introduced in Section 1, devoted to an exposition of Lebesgue results, and is used throughout the paper. Following the norms of the modern mathematical exposition, the continuous narrative of Lebesgue is split into several lemmas and theorems. A similar treatment of the Lusternik–Schnirelmann covering theorems [LS] is the topic of Section 5. In Section 6 these theorems are complemented by several popular and elementary applications, usually going under the name “Borsuk–Ulam theorems”.

Section 2 is devoted to the easier part of Lebesgue results: Lebesgue construction of coverings (also known as tilings) of  $\mathbb{R}^n$  by small closed sets having the property that no more than  $n + 1$  of these sets may have a common point. The usual presentations of Lebesgue construction are limited by a picture illustrating the case  $n = 2$ . In fact, it is much more interesting in higher dimensions. We complement Lebesgue construction by a full description of the intersection pattern of these sets, which, strangely enough, appears to be new.

Section 3 is devoted to Hurewicz's modification [H2] of Lebesgue theory. Hurewicz's proofs bear a substantial similarity to Sperner's ones, but neither simplices, nor the double counting are used. Section 4 is devoted to combinatorics of Hurewicz's results and an application of Hurewicz's theorems to Lebesgue coverings of  $\mathbb{R}^n$ . One would think that this was done around 1930, but, apparently, this is not the case. Surprisingly, this application leads to the strong Kuhn's lemma [Ku] from 1960, which appears in Section 4 as Theorem 4.2.

**Hurewicz and path-following algorithms.** Like Sperner, Hurewicz used the principle “an odd number cannot be equal to 0” in order to prove the existence of desired intersection points. But along the way he constructed a path leading to these points, and his proofs almost explicitly contain a *path-following algorithm* of the variety which became popular after H. Scarf [Sc]. Hurewicz’s path consists of several segments (see Lemma 3.4), and passing from a segment to the next one is nothing else but a *pivot step* of this algorithm. After reading Section 3 all this should be obvious to anyone familiar with such algorithms.

**Cubical cochains and products of cubical cochains.** The most conceptual way to prove covering theorems of Lusternik–Schnirelmann is to use the multiplication in the cohomology rings of the real projective spaces. The multiplication of the relative cohomology classes allows to prove Lebesgue covering theorems in a similar manner.

In Sections 7 and 8 we adapt Serre’s [Se] definition of products of singular cubical cochains to discrete setting and combinatorial cubical cochains. By the reasons partially outlined in Appendix 2 (morally speaking, in order to be in agreement with Poincaré duality), for the discussion of products the solid cubes of Lebesgue are replaced by discrete cubes consisting in dimension  $n$  of  $2^n$  points corresponding to the centers of adjacent Lebesgue cubes.

Products of cubical cochains lead to a new approach to Lebesgue and Lusternik–Schnirelmann covering theorems which is both conceptual and elementary. The main results are new purely combinatorial “cubical lemmas”. Their cochain versions are Theorems 7.4 and 7.6 in Lebesgue context and Theorem 8.4 in Lusternik–Schnirelmann context, and combinatorial versions are Theorems 7.7 and 8.5 respectively. The classical topological covering theorems can be deduced from Theorems 7.7 and 8.5 by standard elementary topological tools. See the proof of Lebesgue first covering theorem in Section 7 and Theorem 8.6.

The little theory developed in Section 7 was partially motivated by the desire to elucidate Kuhn’s cubical Sperner lemma [Ku] with its rather strange, at least to a topologist, statement. See Section 9. It is not hard to translate Kuhn’s lemma into the language of subsets, and then it turns into an immediate corollary of Theorem 7.7. At the same time a weakened version of Kuhn’s lemma, sufficient for all applications, easily follows from Lebesgue results. A similar translation turns Kuhn’s strong lemma into an immediate corollary of Hurewicz’s theorems and combinatorics of Lebesgue tilings. We present also Kuhn’s own proofs, based on Freudenthal’s triangulations (see Appendix 1) of cubes and a Sperner-like argument.

Ky Fan’s cubical Sperner lemma [Fa] appears to be even more mysterious than Kuhn’s one, despite a fairly simple proof. At the first sight it seems that Ky Fan’s *cubical vertex maps* are cubical analogues of simplicial maps, but then it turns out that they are not. By this reason the author took the liberty to rename them as *adjacency-preserving maps*. The property of being adjacency-preserving can be interpreted as a sort of transversality, and the theory of Section 7 allows to understand Ky Fan’s lemma as a natural strengthening of Lebesgue or Kuhn’s results under this transversality assumption. See Section 10.

# 1. Du côté de chez Lebesgue

**The big cube.** Let us fix a natural number  $n$ , and let  $I = \{1, 2, \dots, n\}$ . Let us also fix another natural number  $l$ . The *big cube* of size  $l$  is

$$Q = [0, l]^n \subset \mathbb{R}^n.$$

A *face* of  $Q$  is a product of the form

$$\prod_{i=1}^n J_i,$$

where each  $J_i$  is either the interval  $[0, l]$ , or the one-point set  $\{0\}$ , or  $\{l\}$ . The *dimension* of a face of  $Q$  is defined as the number of intervals in the corresponding product. A face of dimension  $m$  is also called an *m-face*.

For  $i \in I$  let  $\text{pr}_i: Q \rightarrow [0, l]$  be the projection  $(x_1, x_2, \dots, x_n) \mapsto x_i$  and let

$$A_i = \text{pr}_i^{-1}(0) \quad \text{and} \quad B_i = \text{pr}_i^{-1}(l).$$

Clearly,  $A_i$  and  $B_i$  are  $(n-1)$ -faces of  $Q$ , and every  $(n-1)$ -face of  $Q$  is equal to either  $A_i$  or  $B_i$  for some  $i$ . For every  $i \in I$  the faces  $A_i, B_i$  are said to be *opposite*. The usual boundary  $\text{bd}Q$  of  $Q$  is equal to the union of all  $(n-1)$ -faces  $A_i, B_i$ .

**The cubes.** An *n-cube* is a product of the form

$$\prod_{i=1}^n [a_i, a_i + 1],$$

where  $a_1, a_2, \dots, a_n$  are non-negative integers  $\leq l-1$ . The *n-cubes* form a partition of  $Q$  in an obvious sense. A *cubical set* is a subset of  $Q$  equal to a union of *n-cubes*. If  $e$  is a cubical set, then  $\bar{e}$  denotes the union of all *n-cubes* not contained in  $e$ .

A *cube* (which is not necessarily an *n-cube*) is a product of the form

$$c = \prod_{i=1}^n J_i,$$

where for every  $i \in I$  either  $J_i = [a_i, a_i + 1]$  for a non-negative integer  $a_i \leq l-1$ , or the one-point set  $J_i = \{a_i\}$  for a non-negative integer  $a_i \leq l$ . The *dimension* of  $c$  is defined as the number of intervals in the above product. A cube of dimension  $m$  is also called a *m-cube*. A *k-face* of  $c$  is a *k-cube* contained in  $c$ . Every *k-face* of an *m-cube*  $c$  is obtained by replacing in the above product  $m-k$  intervals  $[a_i, a_i + 1]$  by one of their endpoints.

**Cubical chains.** A *cubical chain of dimension  $m$* , or simply an  *$m$ -chain*, is a formal sum of several  $m$ -cubes with coefficients in  $\mathbb{F}_2$ . An  $m$ -chain  $\gamma$  is essentially a set of  $m$ -cubes, namely the set of all  $m$ -cubes entering  $\gamma$  with the coefficient 1. The union of these  $m$ -cubes is called the *support* of  $\gamma$  and is denoted by  $|\gamma|$ . Clearly, an  $m$ -chain  $\gamma$  is equal to the sum of all  $m$ -cubes  $c$  such that  $c \subset |\gamma|$ . We will call such  $m$ -cubes  $c$  the  *$m$ -cubes of  $\gamma$* , or simply the *cubes of  $\gamma$* . Clearly,  $\gamma$  is equal to the sum of the cubes of  $\gamma$ .

The boundary  $\partial c$  of an  $m$ -cube  $c$  is defined as the sum of all its  $(m-1)$ -faces. The *boundary operator*  $\partial$  is the extension of the boundary of  $m$ -cubes to all  $m$ -chains by the linearity. As always, the main property of the boundary operator  $\partial$  is

$$(1) \quad \partial \circ \partial = 0.$$

It is sufficient to check this property for  $m$ -cubes. In this case it follows from the fact that an  $(m-2)$ -face of an  $m$ -cube  $c$  is contained in exactly two  $(m-1)$ -faces of  $c$ .

**Projections.** We need a limited class of maps similar to simplicial maps. Let  $k \leq n$  and let

$$p: [0, l]^n \longrightarrow [0, l]^k$$

be the projection defined by a set of  $k$  coordinates. If  $c$  is an  $m$ -cube, then the image  $p(c)$  is an  $m'$ -cube for some  $m' \leq m$ . For an  $m$ -cube  $c$  let

$$p_*(c) = c \quad \text{if } p(c) \text{ is an } m\text{-cube and}$$

$$p_*(c) = 0 \quad \text{if } p(c) \text{ is an } m'\text{-cube with } m' < m.$$

The map  $p_*$  extends to all  $m$ -chains by linearity. The extended map is also denoted by  $p_*$  and is called the map *induced* by  $p$ . Taking the induced maps respects the compositions in the sense that if

$$r: [0, l]^k \longrightarrow [0, l]^l$$

is a similar projection, then  $(r \circ p)_* = r_* \circ p_*$ , as a trivial verification shows.

**Lemma.** *The map  $p_*$  commutes with  $\partial$  in the sense that for every chain  $\gamma$*

$$(2) \quad \partial \circ p_*(\gamma) = p_* \circ \partial(\gamma).$$

**Proof.** It is sufficient to prove (2) when  $\gamma$  is equal to an  $m$ -cube  $c$ . If  $p(c)$  is also an  $m$ -cube, (2) is obvious. Also, if  $p(c)$  is an  $m'$ -cube with  $m' \leq m-2$ , then both sides of (2)

are equal to 0 and hence (2) holds. It remains to consider the case when  $p(c)$  is an  $(m-1)$ -cube. In this case  $p(d)$  is an  $(m-1)$ -cube for exactly two  $(m-1)$ -faces  $d = d_1, d_2$  of  $c$  and is an  $(m-2)$ -cube for all other  $(m-1)$ -faces  $d$  of  $c$ . Clearly,  $p(d_1) = p(d_2)$  and  $p(d) = 0$  if  $d \neq d_1, d_2$ . Hence this case

$$p_*(\partial c) = p_*(d_1) + p_*(d_2) = 0.$$

Also, in this case  $\partial p_*(c) = \partial 0 = 0$ . Hence the lemma holds in this case also. ■

**Lebesgue construction.** Suppose that  $\gamma$  is an  $m$ -chain and  $e$  is a cubical set. Let  $\gamma_e$  be the sum of all  $m$ -cubes of  $\gamma$  contained in  $e$  and  $\gamma'$  be the sum of all other  $m$ -cubes of  $\gamma$ . Then

$$\gamma = \gamma_e + \gamma'$$

and hence  $\partial\gamma = \partial\gamma_e + \partial\gamma'$ . Clearly,  $|\gamma_e| \subset |\gamma| \cap e$  and  $|\gamma'| \subset |\gamma| \cap \bar{e}$ . Let  $\delta$  be the sum of all common  $(m-1)$ -cubes of the chains  $\partial\gamma_e$  and  $\partial\gamma'$ . Then  $|\delta|$  is contained in both  $|\gamma| \cap e$  and  $|\gamma| \cap \bar{e}$ . Let  $\delta_e$  be the sum of all  $(m-1)$ -cubes of  $\partial\gamma_e$  which are not  $(m-1)$ -cubes of  $\delta$ . Clearly,

$$(3) \quad \partial\gamma_e = \delta_e + \delta.$$

A  $m$ -chain  $\gamma$  is said to be *proper* if  $|\partial\gamma| = |\gamma| \cap \text{bd}Q$  and *special* if it is proper and  $|\gamma|$  is disjoint from  $A_i \cup B_i$  for  $i \leq n - m$ . The following two lemmas play a key role.

**1.1. Lemma.** *If  $\gamma$  is proper, then  $\delta$  is also proper.*

**Proof.** Let us prove first that  $|\partial\delta| \subset \text{bd}Q$ . The  $(m-1)$ -cubes of  $\delta$  are exactly the cubes entering the sum  $\partial\gamma_e + \partial\gamma'$  twice, i.e. cancelling in this sum. The other  $(m-1)$ -cubes in this sum do not cancel and are  $(m-1)$ -cubes of  $\partial\gamma$ . Since  $\gamma$  is proper, these  $(m-1)$ -cubes are contained in  $\text{bd}Q$ . It follows that, in particular,  $|\delta_e| \subset \text{bd}Q$ . Since  $\partial \circ \partial = 0$ , the equality (3) implies that

$$\partial\delta_e + \partial\delta = \partial \circ \partial(\gamma_e) = 0$$

and hence  $\partial\delta = -\partial\delta_e = \partial\delta_e$ . Therefore  $|\partial\delta| = |\partial\delta_e| \subset |\delta_e| \subset \text{bd}Q$  and hence

$$|\partial\delta| \subset \text{bd}Q.$$

Every  $(m-1)$ -cube of  $\delta$  is an  $(m-1)$ -face of one  $m$ -cube of  $\gamma_e$  and one of  $\gamma'$ . Since  $\gamma$  is a proper, these  $m$ -cubes are not contained in  $\text{bd}Q$ . It follows that no  $(m-1)$ -cube of  $\delta$  is contained in  $\text{bd}Q$ . Together with  $|\partial\delta| \subset \text{bd}Q$ , this implies that  $\delta$  is proper. ■

**1.2. Lemma.** Suppose that  $\gamma$  is a special  $m$ -chain. If, moreover,

$$e \cap B_{n-m+1} = \emptyset \quad \text{and} \quad |\gamma| \cap A_{n-m+1} \subset e,$$

then  $|\gamma'|$  is disjoint from  $A_{n-m+1}$  and the  $(m-1)$ -chain  $\delta$  is special.

**Proof.** Let us prove first that  $|\gamma'|$  is disjoint from  $A_{n-m+1}$ . Suppose that  $c$  is an  $m$ -cube of  $\gamma$  intersecting  $A_{n-m+1}$ . Since  $A_{n-m+1}$  is an  $(n-1)$ -face of  $Q$ , in this case either  $c$  is contained in  $A_{n-m+1}$ , or  $c \cap A_{n-m+1}$  is an  $(n-1)$ -face of  $c$ . In the first case

$$c \subset |\gamma| \cap A_{n-m+1} \subset e,$$

and hence  $c$  is a cube of  $\gamma_e$ . In the second case  $c \cap A_{n-m+1} \subset e$  and since  $e$  is a cubical set, the intersection  $c \cap A_{n-m+1}$  is contained in an  $(n-1)$ -face of an  $n$ -cube  $d \subset e$ . It follows that  $c \subset d \subset e$  and hence  $c$  is a cube of  $\gamma_e$  in this case also. It follows that no  $m$ -cube of  $\gamma'$  intersects  $A_{n-m+1}$  and hence  $|\gamma'|$  is disjoint from  $A_{n-m+1}$ .

Since  $|\delta| \subset |\gamma|$ , the support  $|\delta|$  is disjoint from  $A_i \cup B_i$  for  $i \leq n-m$ . Since  $|\delta|$  is contained in  $|\partial\gamma_e| \subset e$  and  $e$  is disjoint from  $B_{n-m+1}$ , the support  $|\delta|$  is disjoint from  $B_{n-m+1}$ . By the construction,  $|\delta| \subset |\partial\gamma'| \subset |\gamma'|$ . By the previous paragraph  $|\gamma'|$  is disjoint from  $A_{n-m+1}$ , and hence  $|\delta|$  is disjoint from  $A_{n-m+1}$  also. ■

**1.3. Corollary.** Suppose that the assumptions of the previous lemma hold. Let  $\beta$  be the sum of all  $(m-1)$ -cubes of  $\partial\gamma_e$  contained in  $A_{n-m+1}$ . Then

$$(4) \quad \partial\gamma_e = \beta + \delta + \delta_0,$$

for some  $(m-1)$ -chain  $\delta_0$  such that  $|\delta_0|$  is contained in  $\bigcup_{k > n-m+1} A_k \cup B_k$ .

**Proof.** By Lemma 1.2 the support  $|\gamma'|$  is disjoint from  $A_{n-m+1}$  and hence an  $(m-1)$ -cube contained in  $A_{n-m+1}$  is a cube of  $\partial\gamma$  if and only if it is a cube of  $\partial\gamma_e$ . Therefore  $\beta$  is equal to the sum of all  $(m-1)$ -cubes of  $\partial\gamma_e$  contained in  $B_{n-m+1}$ .

The support  $|\gamma_e|$  is contained in  $|\gamma|$  and hence is disjoint from  $A_i \cup B_i$  for  $i \leq n-m$ . Also, it is contained in  $e$  and hence is disjoint from  $B_{n-m+1}$ . It follows that every  $(m-1)$ -cube of  $\partial\gamma_e$  is either an  $(m-1)$ -cube of  $\delta$ , or is contained in  $A_{n-m+1}$  and hence is an  $(m-1)$ -cube of  $\beta$ , or is contained in  $A_k \cup B_k$  for some  $k > n-m+1$ . Therefore one can take as  $\delta_0$  the sum of the latter  $(m-1)$ -cubes. ■

**Essential chains.** The *essential chains*, to be defined in a moment, are a tool for proving that some intersections of cubical sets are non-empty. Let us begin with Lebesgue definition (who did not use this term). Let us choose arbitrary non-integer numbers  $a_1, a_2, \dots, a_n \in [0, l]$ .



For an integer  $u$  between 0 and  $n$  let  $L^u$  be the plane in  $\mathbb{R}^n$  defined by the equations

$$x_{u+1} = a_{u+1}, \quad x_{u+2} = a_{u+2}, \quad \dots, \quad x_n = a_n,$$

where  $(x_1, x_2, \dots, x_n)$  are the coordinates in  $\mathbb{R}^n$ . In particular,  $L^u$  is a  $u$ -dimensional plane,  $L^0$  consists of a single point, namely  $(a_1, a_2, \dots, a_n)$ , and  $L^n = \mathbb{R}^n$ . The numbers  $a_i$  are chosen to be non-integers in order to ensure the following property: if  $c$  is an  $(n - u)$ -cube, then the intersection  $c \cap L^u$  is either empty, or consists of a single point. More generally, if  $c$  is an  $(n - k)$ -cube, then  $c \cap L^u$  is either empty, or a cube of dimension  $u - k$  in the usual geometric sense (of course, this intersection is not a cube according to our definition). Using the modern language, one can say that all cubes from our partition of  $Q$  are in *general position* with respect to the planes  $L^u$ .

Now we are ready for the definition. An  $m$ -chain  $\gamma$  is said to be *essential* if  $\gamma$  is special and

$$|\gamma| \cap L^{n-m}$$

consists of an odd number of points. In particular, if  $\gamma$  is essential, then  $|\gamma|$  is non-empty. In Lebesgue terminology a set of the form  $|\gamma|$ , where  $\gamma$  is an essential  $(n - p)$ -chain, is called a set (of the type)  $J_p$ . This is a central notion of his paper [L2].

An obvious drawback of this definition is the dependence on the choice of the numbers  $a_i$ . In fact, the property of being an essential chain does not depend on this choice and Lebesgue arguments work for any fixed choice. In the context of Lebesgue papers [L1], [L2] there is another drawback, the dependence on geometric ideas. After reliance on geometric language rendered to be unconvincing Lebesgue outline [L1] of his ideas about the invariance of the dimension, he decided to “d’arithmétiser la démonstration”. While Lebesgue arguments in [L2] are complete and never met any objection, the following definition is essentially a combinatorial one and allows to complete Lebesgue plan of “arithmetization” of his arguments.

For every  $m \in I$  let  $P_m = A_1 \cap A_2 \cap \dots \cap A_{n-m}$ . Clearly,  $P_m$  is an  $m$ -face of  $Q$ . Let

$$p_m: Q \longrightarrow P_m$$

be the canonical projection. An  $m$ -chain  $\gamma$  is said to be a *essential* if it is special and

$$p_{m*}(\gamma) = \llbracket P_m \rrbracket,$$

where  $\llbracket P_m \rrbracket$  is the sum of all  $m$ -cubes contained in  $P_m$ . Clearly, if  $\gamma$  is essential, then  $\gamma \neq 0$  and hence  $|\gamma| \neq \emptyset$ . The main result about essential chains is Lemma 1.4 below, in which one can understand the term “essential” in either of the above two ways. We will give two proofs of this lemma, the first one is following Lebesgue [L2], while the second one is based on the second definition. We will not use the equivalence of these two definitions of essential chains and leave the proof of their equivalence as an exercise to the reader.

**Intersections of chains with the planes  $L^u$ .** The first proof requires some additional preparations. For an integer  $u$  between 0 and  $n$  let  $Q^u = Q \cap L^u$ . Clearly,

$$Q^u = [0, l]^u \times (a_{u+1}, a_{u+2}, \dots, a_n).$$

Equivalently,  $Q^u$  is the  $u$ -dimensional cube  $[0, l]^u \subset \mathbb{R}^u \subset \mathbb{R}^n$  shifted by the vector

$$(0, 0, \dots, 0, a_{u+1}, a_{u+2}, \dots, a_n) \in \mathbb{R}^n.$$

This allows to transfer to  $Q^u$  the notions of cubes, chains, etc. Moreover, we can take the intersection of chains in  $Q$  with  $Q^u$ . Namely, let  $\gamma$  be an  $(n-k)$ -chain in  $Q$ , i.e. let

$$\gamma = \sum_i c_i,$$

where  $c_i$  are  $(n-k)$ -cubes in  $Q$ . The fact that all cubes in  $Q$  are in general position with respect to  $L^u$  allows to define the *intersection*  $\gamma \cap L^u$  as the  $(u-k)$ -chain

$$\gamma \cap L^u = \sum_i c_i \cap L^u,$$

where the empty intersections are interpreted as 0's. The operation of taking the intersection of chains with  $L^u$  commutes with  $\partial$ , i.e.

$$(\partial\gamma) \cap L^u = \partial(\gamma \cap L^u)$$

for any chain  $\gamma$  in  $Q$ . This is trivial for cubes, and extends to the general case by linearity.

**1.4. Lemma.** *Under the assumptions of Lemma 1.2, if the chain  $\gamma$  is essential, then the chain  $\delta$  is also essential.*

**Proof based on intersections.** In this proof we use Lebesgue definition of essential chains. The idea is to consider the intersections of chains with  $L^{n-m+1}$ . For a chain  $\varepsilon$  in  $Q$  let

$$\cap\varepsilon = \varepsilon \cap L^{n-m+1}.$$

Then  $\cap\gamma$  and  $\cap\gamma_e$  are 1-chains and  $\cap\delta$  is a 0-chain in  $Q^{n-m+1}$ . By the definition,  $\delta$  is essential if and only if  $|\cap\delta|$  consists of an odd number of points. Since  $\gamma$  and  $\delta$  are proper and  $\partial$  commutes with taking intersections,  $\cap\gamma$  and  $\cap\delta$  are proper as chains in  $Q^{n-m+1}$ . Since  $\gamma$  and  $\delta$  are special, this implies that  $\cap\gamma$  and  $\cap\delta$  are special as chains in  $Q^{n-m+1}$  and hence  $|\cap\gamma|$  is disjoint from all  $(n-m)$ -faces of  $Q^{n-m+1}$  except

$$A_{n-m+1} \cap Q^{n-m+1} \quad \text{and} \quad B_{n-m+1} \cap Q^{n-m+1},$$

and  $|\cap\delta|$  is disjoint from the whole boundary  $\text{bd } Q^{n-m+1}$ .

Let us consider now the chains  $\beta$  and  $\delta_0$  from Corollary 1.3. The intersection  $\delta_0 \cap L^{n-m+1}$  is empty and hence  $\cap \delta_0 = 0$ . By intersecting both sides of (4) with  $L^{n-m+1}$  we see that

$$\partial(\cap \gamma_e) = \cap \partial \gamma_e = \cap \beta + \cap \delta + \cap \delta_0 = \cap \beta + \cap \delta.$$

The boundary of every 1-chain is a sum of an even numbers of points. It follows that  $\cap \delta$  is a sum of an odd number of points if and only if  $\cap \beta$  is.

Since  $\gamma$  is essential,  $|\gamma| \cap L^{n-m}$  consists of an odd number of points. But  $|\gamma| \cap L^{n-m}$  is equal to  $|\cap \gamma| \cap Q^{n-m}$ . It follows that the number of 1-cubes of  $\cap \gamma$  intersecting  $Q^{n-m}$  is odd. Let  $\eta$  be the sum of these 1-cubes and  $\theta$  be the sum of 1-cubes of  $\cap \gamma$  contained in

$$Q^{n-m} = [0, l]^{n-m} \times [0, a_{n-m+1}] \times (a_{n-m+2}, \dots, a_n).$$

By the construction, out of two endpoints of a 1-cube of  $\eta$  one is in  $Q^{n-m}$  and the other is not. Let  $\alpha$  be the sum of the endpoints which are not in  $Q^{n-m}$ . The number of them is equal to the number of 1-cubes of  $\eta$  and hence is odd. Since  $\gamma$  is a proper chain,

$$\partial(\eta + \theta) = \alpha + \beta.$$

The chain  $\partial(\eta + \theta)$  is a boundary and hence is a sum of an even number of points. Since  $\alpha$  is a sum of an odd number of points, it follows that  $\beta$  is a sum of an odd number of points. By the previous paragraph, this implies that  $\cap \delta$  is a sum of an odd number of points. Some of these points cancel under the summation, but this does not affect the parity of their number. Therefore  $|\cap \delta|$  consists of an odd number of points and  $\delta$  is essential. ■

**Proof based on projections.** Let  $p = p_m$ . Since  $\gamma$  is essential,  $p_*(\gamma) = \llbracket P_m \rrbracket$  and

$$p_*(\partial \gamma) = \partial p_*(\gamma) = \partial \llbracket P_m \rrbracket,$$

as follows from (2). Since  $A_{n-m+1} = p^{-1}(P_{m-1})$ , this implies that

$$p_*(\beta) = \llbracket P_{m-1} \rrbracket,$$

where  $\beta$  is the chain from Corollary 1.3. Next, let  $q = p_{m-1}$ . Then  $q = q \circ p$  and hence

$$q_*(\beta) = q_*(p_*(\beta)) = q_*(\llbracket P_{m-1} \rrbracket).$$

Since  $q$  is equal to the identity on  $P_{m-1}$ , this implies that

$$q_*(\beta) = \llbracket P_{m-1} \rrbracket.$$

Therefore, in order to prove that  $\delta$  is essential it is sufficient to prove that  $q_*(\delta) = q_*(\beta)$ .

By applying  $q_*$  to (4) we see that

$$q_*(\beta) + q_*(\delta) + q_*(\delta_0) = q_*(\partial\gamma_E) = \partial p_*(\gamma_E).$$

But  $q$  takes every  $m$ -cube to an  $m'$ -cube with  $m' \leq m - 1$ . Since  $\gamma_E$  is an  $m$ -chain, it follows that  $q_*(\gamma_E) = 0$  and hence  $\partial q_*(\gamma_E) = 0$ . Therefore

$$q_*(\beta) + q_*(\delta) + q_*(\delta_0) = 0.$$

Also, if an  $(m - 1)$ -cube  $c$  is contained in  $A_k$  or  $B_k$  with  $k > n - m + 1$ , then  $q(c)$  is an  $m'$ -cube with  $m' \leq m - 2$ . It follows that  $q_*(\delta_0) = 0$  and therefore

$$q_*(\beta) + q_*(\delta) = 0,$$

or, equivalently,  $q_*(\delta) = q_*(\beta)$ . As we saw, this implies that  $\delta$  is essential. ■

**1.5. Theorem.** *Suppose that  $e_1, e_2, \dots, e_{n+1}$  are cubical sets covering  $Q$  and such that*

- (i)  $e_1 \cup e_2 \cup \dots \cup e_i$  contains  $A_i$  and  $e_i$  is disjoint from  $B_i$  for  $i \leq n$ , and
- (ii)  $e_i$  is disjoint from  $A_j$  if  $i > j$ .

*Then the intersection of the sets  $e_1, e_2, \dots, e_{n+1}$  is non-empty.*

**Proof.** We will recursively construct essential  $(n - m)$ -chains  $\gamma_m$  in  $Q$  such that

$$|\gamma_m| \subset e_1 \cap e_2 \cap \dots \cap e_m$$

and

$$(5) \quad |\gamma_m| \subset e_{m+1} \cup e_{m+2} \cup \dots \cup e_{n+1}.$$

Let  $\gamma_0 = \llbracket Q \rrbracket$ , where  $\llbracket Q \rrbracket$  is the sum of all  $n$ -cubes of  $Q$ . Clearly, the  $n$ -chain  $\gamma_0$  is essential and the two other conditions are vacuous for  $m = 0$ . Suppose that  $\gamma_m$  is already constructed. Let us apply Lebesgue construction to  $\gamma = \gamma_m$  and  $e = e_{m+1}$ , and let  $\delta$  be the resulting  $(n - m - 1)$ -chain. By Lemma 1.1 the chain  $\delta$  is proper.

Let us check the assumptions of Lemma 1.2. The assumption (i) of the theorem implies that  $e_{m+1} \cap B_{m+1} = \emptyset$ . The assumption (ii) of the theorem implies that  $e_{m+2} \cup \dots \cup e_{n+1}$  is disjoint from  $A_{m+1}$  and hence (5) implies that  $|\gamma_m| \cap A_{m+1}$  is contained in  $e_{m+1}$ . Therefore the assumptions of Lemma 1.2 hold and this lemma implies the chain  $\delta$  is special. Now we can apply Lemma 1.4 and conclude that the chain  $\delta$  is essential.

By the construction  $|\delta|$  is contained in  $|\gamma_m| \cap e_{m+1}$  and hence

$$|\delta| \subset e_1 \cap e_2 \cap \dots \cap e_{m+1}.$$

Also by the construction, every cube of  $\delta$  is a common face of two cubes of  $\gamma_m$ , one of which is contained in  $e_{m+1}$  and the other does not. By (5) that other cube is contained in  $e_{m+2} \cup \dots \cup e_{n+1}$ . It follows that

$$|\delta| \subset e_{m+2} \cup \dots \cup e_{n+1}.$$

Therefore we can set  $\gamma_{m+1} = \delta$ . This completes the construction of the chains  $\gamma_m$ .

Let us consider the last of these chains, namely  $\gamma_n$ . The support  $|\gamma_n|$  is contained in the intersection  $e_1 \cap e_2 \cap \dots \cap e_n$  and by (5) also in  $e_{n+1}$ . Since  $\gamma_n$  is essential and hence  $|\gamma_n| \neq \emptyset$ , this implies that  $e_1 \cap e_2 \cap \dots \cap e_{n+1} \neq \emptyset$ . ■

**Lebesgue fusion of sets.** Let  $d_1, d_2, \dots, d_r$  be a covering of  $Q$  by several sets (in the applications these sets are closed or even cubical). Suppose that none of the sets  $d_i$  intersects two opposite  $(n-1)$ -faces of  $Q$ . An elementary construction of Lebesgue allows to construct sets  $e_1, e_2, \dots, e_{n+1}$  covering  $Q$  and satisfying the conditions (i) and (ii) of Theorem 1.5 by taking unions of disjoint collections of sets  $d_i$ .

Let  $e_1$  be the union of all sets  $d_j$  having non-empty intersection with  $A_1$ . Then  $e_1$  is disjoint from  $B_1$ . Let  $e_2$  be the union of all sets  $d_j$  which have non-empty intersection with  $A_2$  and were not used to form  $e_1$ . Since the sets  $d_i$  form a covering of  $Q$ , the set  $A_2$  is contained in  $e_1 \cup e_2$ . Clearly,  $e_2$  is disjoint from  $A_1$  and  $B_2$ .

Suppose that  $k \leq n-1$ , the sets  $e_1, e_2, \dots, e_k$  are already defined and satisfy the conditions (i) and (ii) of Theorem 1.5 for  $i \leq k$ . Let  $e_{k+1}$  be the union of all sets  $d_j$  which have non-empty intersection with  $A_{k+1}$  and were not used to form sets  $e_1, e_2, \dots, e_k$ . By the construction,  $e_{k+1}$  satisfies the conditions (i) and (ii) of Theorem 1.5 for  $i = k+1$ .

Finally, let  $e_{n+1}$  be the union of the sets  $d_j$  not used to form  $e_1, e_2, \dots, e_n$ . The condition (i) is vacuous for  $i = n+1$ , and  $e_{n+1}$  obviously satisfies the condition (ii) for  $i = n+1$ .

**1.6. Theorem.** *Let  $d_1, d_2, \dots, d_r$  be a covering of  $Q$  by cubical sets. Suppose that none of the sets  $d_i$  intersects two opposite  $(n-1)$ -faces of  $Q$ . Then among the sets  $d_i$  there are  $n+1$  sets with non-empty intersection.*

**Proof.** Let  $e_1, e_2, \dots, e_{n+1}$  be the sets obtained from  $d_1, d_2, \dots, d_r$  by Lebesgue fusion construction. Then the assumptions of Theorem 1.5 hold and hence  $e_1 \cap e_2 \cap \dots \cap e_{n+1}$  is non-empty. Since the sets  $e_i$  are unions of disjoint collections of sets  $d_j$ , this implies that among the sets  $d_i$  there are  $n+1$  sets with non-empty intersection. ■

**Lebesgue first covering theorem.** Let  $D_1, D_2, \dots, D_r$  be a covering of the unit cube  $[0, 1]^n$  by closed sets. Suppose that none of the sets  $D_i$  intersects two opposite  $(n-1)$ -faces of  $[0, 1]^n$ . Then among the sets  $D_i$  there are  $n+1$  sets with non-empty intersection.

**Proof.** Let  $\varepsilon > 0$  be a Lebesgue number of the covering of the unit cube  $[0, 1]^n$  by the sets  $D_1, D_2, \dots, D_r$  and  $(n-1)$ -faces of  $[0, 1]^n$ . Suppose that the number  $l$  is chosen to be so large that an  $n$ -dimensional cube in  $\mathbb{R}^n$  with the sides of the length  $1/l$  has diameter  $< \varepsilon$ . Let  $p: Q \rightarrow [0, 1]^n$  be the map defined by

$$p(x_1, x_2, \dots, x_n) = (x_1/l, x_2/l, \dots, x_n/l).$$

For each  $i \leq r$  let  $d_i$  be the union of all  $n$ -cubes of  $Q$  intersecting the preimage  $p^{-1}(D_i)$ . The sets  $d_1, d_2, \dots, d_r$  are cubical sets. Obviously, they cover  $Q$ . If a cube  $d_i$  intersects an  $(n-1)$ -face of  $Q$ , then  $D_i$  intersects the image of this face in  $[0, 1]^n$  (in fact, the  $(n-1)$ -faces of  $[0, 1]^n$  were included in the covering defining  $\varepsilon$  precisely to ensure this property). It follows that none of the sets  $d_i$  intersects two opposite  $(n-1)$ -faces of  $Q$ . By Theorem 1.6 among these cubical sets there are  $n+1$  sets with non-empty intersection. If, say,  $d_1 \cap d_2 \cap \dots \cap d_{n+1}$  is non-empty and  $x \in d_1 \cap d_2 \cap \dots \cap d_{n+1}$ , then the distance of the point  $p(x) \in [0, 1]^n$  from each of the sets  $D_1, D_2, \dots, D_{n+1}$  is  $< \varepsilon$ . Now Lebesgue lemma for closed sets implies that the intersection  $D_1 \cap D_2 \cap \dots \cap D_{n+1}$  is non-empty. ■

**Lebesgue second covering theorem.** If  $\varepsilon > 0$  is sufficiently small, then every closed  $\varepsilon$ -covering of  $[0, 1]^n$  has order  $\geq n+1$ .

**Proof.** It is sufficient to note that no set of diameter  $< 1$  can intersect two opposite  $(n-1)$ -faces of  $[0, 1]^n$  and hence one can simply take  $\varepsilon = 1$ . ■

**Further applications.** The above results are contained in Lebesgue papers [L1], [L2] either explicitly or implicitly. Now we will present some further applications of Theorem 1.5.

**1.7. Theorem.** Let  $d_1, d_2, \dots, d_{n+1}$  be cubical sets covering  $Q$  and such that the union  $d_1 \cup d_2 \cup \dots \cup d_i$  contains  $A_i$  and  $d_i$  is disjoint from  $B_i$  for every  $i \leq n$ . Then the intersection  $d_1 \cap d_2 \cap \dots \cap d_{n+1}$  is non-empty.

**Proof.** For  $i \leq n+1$  let  $e_i$  be the union of all  $n$ -cubes contained in  $d_i$ , but not in any  $d_j$  with  $j < i$ . Then  $e_i$  is disjoint from  $A_j$  if  $i > j$ , and

$$e_1 \cup e_2 \cup \dots \cup e_i = d_1 \cup d_2 \cup \dots \cup d_i$$

for every  $i \leq n$ . It follows that  $e_1, e_2, \dots, e_{n+1}$  satisfy the assumptions of Theorem 1.5 and hence the intersection  $e_1 \cap e_2 \cap \dots \cap e_{n+1}$  is non-empty. Since  $e_i \subset d_i$  for every  $i$ , it follows that the intersection  $d_1 \cap d_2 \cap \dots \cap d_{n+1}$  is non-empty. ■

**1.8. Theorem.** Let  $D_1, D_2, \dots, D_{n+1}$  be closed sets covering  $[0, 1]^n$  and such that the union  $D_1 \cup D_2 \cup \dots \cup D_i$  contains  $A_i$  and  $D_i$  is disjoint from  $B_i$  for every  $i \leq n$ . Then the intersection  $D_1 \cap D_2 \cap \dots \cap D_{n+1}$  is non-empty.

**Proof.** The proof is completely similar to the proof of Lebesgue first covering theorem, with Theorem 1.7 playing the role of Theorem 1.6. ■

**1.9. Theorem.** Let  $D_1, D_2, \dots, D_{n+1}$  be closed sets covering  $[0, 1]^n$  and such that the set  $D_k$  is disjoint from  $A_i$  with  $i < k$  for every  $k \leq n+1$  and is disjoint from  $B_k$  for every  $k \leq n$ . Then the intersection  $D_1 \cap D_2 \cap \dots \cap D_{n+1}$  is non-empty.

**Proof.** For every  $i \leq n$ , the union  $D_1 \cup D_2 \cup \dots \cup D_{n+1}$  contains  $A_i$  and hence the assumptions of the theorem imply that  $D_1 \cup D_2 \cup \dots \cup D_i$  contains  $A_i$ . Hence Theorem 1.8 applies to the sets  $D_1, D_2, \dots, D_{n+1}$  and implies that their intersection is non-empty. ■

**Separation of faces.** Let  $A, B$  be two disjoint subsets of a closed subset  $X \subset \mathbb{R}^n$  (or a topological space  $X$ ). A closed subset  $C \subset X$  is said to be a *partition between  $A$  and  $B$  in  $X$*  if  $A$  and  $B$  are contained in two different components of the complement  $X \setminus C$ . The next theorem is a basic result of dimension theory. See, for example, [HW], Section IV.1. It is concerned with sets separating  $A_i$  from  $B_i$  in  $Q$  in the case  $l = 1$ , i.e.  $Q = [0, 1]^n$ .

**Theorem about partitions.** Let  $C_1, C_2, \dots, C_n$  be closed subsets of  $[0, 1]^n$  such that for every  $i$  the set  $C_i$  is a partition between  $A_i$  and  $B_i$ . Then  $C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset$ .

**Proof.** Let  $U_i$  be the component of  $[0, 1]^n \setminus C_i$  containing  $A_i$ , and let  $D_i$  be its closure. Let  $E_i$  be the closure of  $[0, 1]^n \setminus D_i$ . Then  $B_i$  is contained in  $E_i$ . If  $x \in [0, 1]^n$  is not in  $C_i$ , then either  $x \in U_i$ , or  $x$  belongs to some other component of  $[0, 1]^n \setminus C_i$ . In the first case  $x \notin E_i$ , in the second case  $x \notin D_i$ . It follows that  $D_i \cap E_i \subset C_i$ . Let  $D_{n+1} = E_1 \cap E_2 \cap \dots \cap E_n$ . The sets  $D_1, D_2, \dots, D_{n+1}$  form a closed covering of  $[0, 1]^n$  satisfying the assumptions of Theorem 1.8 and hence their intersection is non-empty. Let  $x$  be a point in the intersection of these sets. Then

$$x \in D_i \cap D_{n+1} \subset D_i \cap E_i \subset C_i$$

for every  $i$ . It follows that  $C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset$ . ■

**Two cartesian versions of KKM argument.** Suppose that  $f: [0, 1]^n \rightarrow [0, 1]^n$  is a continuous map without fixed points. By slightly perturbing  $f$ , if necessary, we may assume that the image of  $f$  is disjoint from  $\text{bd}[0, 1]^n$ . For  $x \in [0, 1]^n$  let  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  be the usual coordinates of  $x$  and  $y = f(x)$  respectively.

**Deducing Brouwer fixed point theorem from Theorem 1.9.** For each  $i = 1, 2, \dots, n$  let  $D_i \subset [0, 1]^n$  be the set of points  $x$  such that  $y_i \geq x_i$ . Let  $D_{n+1} \subset [0, 1]^n$  be the set of points  $x$  such that  $y_i \leq x_i$  for every  $i = 1, 2, \dots, n$ .

If  $x \in A_i$ , then  $x_i = 0$  and  $y_i > 0 = x_i$  because the image of  $f$  is disjoint from  $A_i$  by the above assumption. It follows that  $A_i$  is disjoint from  $D_{n+1}$  for every  $i \leq n$ . If  $x \in B_i$ , then  $x_i = 1$  and  $y_i < 1 = x_i$  because  $B_i$  is disjoint from the image of  $f$  by the above assumption. It follows that  $D_i$  is disjoint from  $B_i$ .

Therefore Theorem 1.9 applies and hence there exists a point  $x \in D_1 \cap D_2 \cap \dots \cap D_{n+1}$ . Now  $x \in D_i$  implies that  $y_i \geq x_i$  and  $x \in D_{n+1}$  implies that  $y_i \leq x_i$  for every  $i$ . It follows that  $y = x$ , i.e.  $x$  is a fixed point of  $f$ . ■

**Deducing Brouwer fixed point theorem from Theorem about partitions.** Let  $C_i$  be the set of points  $x$  such that  $y_i = x_i$ . The complement  $[0, 1]^n \setminus C_i$  is equal to the union of two disjoint open sets, one defined by the inequality  $y_i > x_i$  and another by the inequality  $y_i < x_i$ . Since the image of  $f$  is disjoint from the boundary  $\text{bd}[0, 1]^n$ , the face  $A_i$  is contained in the first of these open sets and the face  $B_i$  in the second. Therefore  $C_i$  is a partition between  $A_i$  and  $B_i$  and hence  $C_1 \cap C_2 \cap \dots \cap C_n$  is non-empty. If  $x$  belongs to this intersection, then  $y_i = x_i$  for all  $i$  and hence  $y = x$ , i.e.  $f(x) = x$ . ■

**The unit cube and the standard simplex.** Let  $\delta$  be the standard  $n$ -simplex in  $\mathbb{R}^{n+1}$  defined in the standard coordinates  $y_0, y_1, \dots, y_n$  by the conditions

$$y_0 + y_1 + \dots + y_n = 1 \quad \text{and} \quad y_0, y_1, \dots, y_n \geq 0,$$

and let  $\delta_i$  be the face of  $\delta$  defined by the equation  $y_i = 0$ , where  $i = 0, 1, \dots, n$ . Let

$$\lambda: [0, 1]^n \longrightarrow \delta$$

be the map defined by  $\lambda(x_1, x_2, \dots, x_n) = (y_0, y_1, \dots, y_n)$ , where

$$y_0 = 1 - x_1,$$

$$y_i = (1 - x_{i+1}) x_1 x_2 \dots x_i \quad \text{for} \quad 0 < i < n,$$

$$y_n = x_1 x_2 \dots x_n.$$

Clearly,

$$y_0 + y_1 + \dots + y_i = 1 - x_1 x_2 \dots x_{i+1}$$

for  $i < n$  and hence  $y_0 + y_1 + \dots + y_n = 1$ .



The preimages under the map  $\lambda$  of the faces  $\delta_i$  are the following:

$$\begin{aligned} \lambda^{-1}(\delta_0) &= B_1, \\ (6) \quad \lambda^{-1}(\delta_i) &= B_{i+1} \cup A_1 \cup A_2 \cup \dots \cup A_i \quad \text{for } 0 < i < n, \quad \text{and} \\ \lambda^{-1}(\delta_n) &= A_1 \cup A_2 \cup \dots \cup A_n. \end{aligned}$$

**Sperner's version of Lebesgue first covering theorem.** *Suppose that  $F_0, F_1, \dots, F_n$  is a closed covering of  $\delta$  such that  $F_i$  is disjoint from  $\delta_i$  for all  $i$ . Then the intersection of the sets  $F_0, F_1, \dots, F_n$  is non-empty.*

**Proof.** Let  $D_i = \lambda^{-1}(F_{i-1})$  for every  $i = 1, 2, \dots, n+1$ . Then  $D_1, D_2, \dots, D_{n+1}$  is a closed covering of  $[0, 1]^n$ . Since  $F_i$  is disjoint from  $\delta_i$  for every  $i$ , (6) implies that

$$\begin{aligned} D_{n+1} &\text{ is disjoint from } A_1 \cup A_2 \cup \dots \cup A_n, \quad \text{and} \\ D_k &\text{ is disjoint from } B_k \text{ and } A_i \text{ with } i < k \end{aligned}$$

for  $k \leq n$ . Therefore the sets  $D_1, D_2, \dots, D_{n+1}$  satisfy the assumptions of Theorem 1.9 and hence  $D_1 \cap D_2 \cap \dots \cap D_{n+1} \neq \emptyset$ . But if  $x \in D_1 \cap D_2 \cap \dots \cap D_{n+1}$ , then  $\lambda(x) \in F_i$  for every  $i$  and hence the intersection of the sets  $F_0, F_1, \dots, F_n$  is non-empty. ■

**Knaster–Kuratowski–Mazurkiewicz and Lebesgue paper.** Knaster, Kuratowski, and Mazurkiewicz were aware of Lebesgue paper [L2], but in [KKM] this paper is only mentioned. But they included Theorem 1.9 in [KKM], referring to W. Hurewicz [H1], [H2]. W. Hurewicz stated Theorem 1.9 in a footnote in [H1] as a theorem of Lebesgue and Brouwer, and referred to [H2] for this particular form of their results. As we saw, the main idea of Knaster–Kuratowski–Mazurkiewicz argument can be easily adapted to use Theorem 1.9 instead of KKM theorem. But Knaster, Kuratowski, and Mazurkiewicz missed this opportunity to use more familiar cartesian coordinates instead of barycentric ones. They went in the opposite direction and deduced Theorem 1.9 from Sperner's version of Lebesgue theorems. It seems that H.W. Kuhn [Ku] was the first to adapt the KKM argument to cartesian coordinates.

Here is their deduction. If the sets  $D_1, D_2, \dots, D_{n+1}$  satisfy the assumptions of Theorem 1.9, then  $F_i = \lambda(D_{i+1})$  is disjoint from  $\delta_i$  for every  $i = 0, 1, \dots, n$  and  $F_0, F_1, \dots, F_n$  cover  $\delta$ . By Sperner's version of Lebesgue first covering theorem  $F_0 \cap F_1 \cap \dots \cap F_n \neq \emptyset$ . If  $x$  belongs to this intersection, then  $x$  cannot belong to any face  $\delta_i$  and hence belongs to the interior of  $\delta$ . But  $\lambda$  induces a bijection between the interiors of  $[0, 1]^n$  and  $\delta$ . It follows that  $\lambda^{-1}(x)$  belongs to the intersection  $D_1 \cap D_2 \cap \dots \cap D_{n+1}$ , which is therefore non-empty. Probably, this deduction [KKM] is the earliest appearance of the map  $\lambda$ . About two decades later the map  $\lambda$  was rediscovered by Serre [Se].

## 2. Lebesgue tilings

**Lebesgue coverings of order  $n + 1$ .** In order to prove the topological invariance of dimension, the second covering theorem of Lebesgue should be complemented by a construction of  $\varepsilon$ -coverings of the cube  $[0, 1]^n$  of order  $n + 1$  for every  $\varepsilon > 0$ . Actually, Lebesgue constructed coverings of order  $n + 1$  of the whole  $\mathbb{R}^n$  by cubes with the side  $\varepsilon'$  for every  $\varepsilon' > 0$ . Moreover, his coverings are *partitions*, i.e. consist of cubes with disjoint interiors. We will call them *Lebesgue tilings*. The second Lebesgue covering theorem is often called *Lebesgue tiling theorem* (it claims that arbitrary closed  $\varepsilon$ -coverings behave like these tilings).

We will construct Lebesgue tilings only for  $\varepsilon' = 1$ . Obviously, they can be scaled to make the cubes of scaled partitions arbitrarily small.

Let us consider for every  $n$  real numbers  $a_1, a_2, \dots, a_n \in \mathbb{R}$  the cube

$$c(a_1, a_2, \dots, a_n) = \prod_{i=1}^n [a_i, a_i + 1].$$

Clearly, the cubes  $c(a_1, a_2, \dots, a_n)$  with integer  $a_1, a_2, \dots, a_n$  form a partition of  $\mathbb{R}^n$ , but its order  $2^n$  is too big. Lebesgue idea is to translate these cubes in the directions of coordinate axes. Let us fix  $n - 1$  real numbers  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1} \in \mathbb{R}$  and consider the cubes

$$e(a_1, a_2, \dots, a_n) = c(u_1, u_2, \dots, u_n),$$

where

$$\begin{aligned} u_1 &= a_1 + a_2 \varepsilon_1 + a_3 \varepsilon_2 + \dots + a_n \varepsilon_{n-1}, \\ u_2 &= a_2 + a_3 \varepsilon_2 + \dots + a_n \varepsilon_{n-1}, \\ (7) \quad &\dots\dots\dots \end{aligned}$$

$$u_{n-1} = a_{n-1} + a_n \varepsilon_{n-1},$$

$$u_n = a_n,$$

and  $a_1, a_2, \dots, a_n$  are integers.

The collection of cubes  $e(a_1, a_2, \dots, a_n)$  can be constructed recursively starting with the partition of  $\mathbb{R}$  into intervals  $[a, a + 1]$ , where  $a$  is an integer. Let us consider  $\mathbb{R}^n$  as the union of the *layers*  $\mathcal{L}(a) = \mathbb{R}^{n-1} \times [a, a + 1]$ , where  $a$  is an integer. Then the cube  $e(a_1, a_2, \dots, a_n)$  is contained in the layer  $\mathcal{L}(a_n)$ . The cubes contained in the layer  $\mathcal{L}(0)$  are the products of similar cubes in  $\mathbb{R}^{n-1}$  (constructed using  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}$ ) with the interval  $[0, 1]$ . The cubes in the layer  $\mathcal{L}(a_n)$  are obtained from the cubes in the layer  $\mathcal{L}(0)$  by a translation along the  $n$ th coordinate axis moving this layer to the layer  $\mathcal{L}(a_n)$  followed

by a translation inside the layer  $\mathcal{L}(a_n)$  by the vector

$$(a_n \varepsilon_{n-1}, a_n \varepsilon_{n-1}, \dots, a_n \varepsilon_{n-1}, 0).$$

In particular, we see that the cubes  $e(a_1, a_2, \dots, a_n)$  indeed form a partition of  $\mathbb{R}^n$ . It turns out that the order of this partition is equal to  $n + 1$  under some mild assumptions about the numbers  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$ . Let us say that the numbers  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  are *generic* if for every  $i \leq n - 1$  the number  $\varepsilon_i$  is not equal to any linear combination

$$m_0 + m_1 \varepsilon_1 + m_2 \varepsilon_2 + \dots + m_{i-1} \varepsilon_{i-1}$$

of the numbers  $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{i-1}$  with integer coefficients  $m_0, m_1, m_2, \dots, m_{i-1}$ . For example, Lebesgue choice  $\varepsilon_i = 2^{-i}$  is generic.

**2.1. Theorem.** *If the choice of numbers  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  is generic, then the intersection of every  $r$  different cubes of the form  $e(a_1, a_2, \dots, a_n)$ , where  $a_1, a_2, \dots, a_n$  are integers, is contained a plane of dimension  $n + 1 - r$  defined by  $r - 1$  equations of the form*

$$x_i = A_i,$$

where  $A_i$  is a linear combination of the numbers  $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  with integer coefficients. In particular, every  $n + 2$  different cubes have empty intersection and the order of the covering of  $\mathbb{R}^n$  by these cubes is equal to  $n + 1$ .

**Proof.** The theorem is obvious for  $n = 1$ . Suppose that the theorem is true with  $n - 1$  in the role of  $n$ . Then the intersections of cubes contained in a single layer  $\mathcal{L}(a_n)$  have the required property. In more details, the intersections of any  $r$  different cubes in the layer  $\mathcal{L}(a_n)$  is contained in a plane defined by equations of the form

$$(8) \quad x_i = A_i + a_n \varepsilon_{n-1},$$

where  $A_i$  is a linear combination of  $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}$  with integer coefficients. Suppose that our  $r$  cubes contained in two layers one next to the other (otherwise their intersection is empty), say, in the layers  $\mathcal{L}(a_n - 1)$  and  $\mathcal{L}(a_n)$ , but not in one of them. Suppose that  $r_0$  of these cubes are contained in  $\mathcal{L}(a_n - 1)$  and  $r_1$  in  $\mathcal{L}(a_n)$ . Then  $r = r_0 + r_1$  and  $r_0, r_1 > 0$ . Clearly, the intersection of these  $r$  cubes is contained in the hyperplane  $\mathbb{R}^{n-1} \times a_n$ . By the previous paragraph, the intersection of  $r_1$  cubes in  $\mathcal{L}(a_n)$  is contained in plane defined by the equations of the form (8) with  $i \neq n$ , and the intersection of  $r_0$  cubes in  $\mathcal{L}(a_n - 1)$  is contained in a plane with the equations of the form

$$x_j = A'_j + (a_n - 1) \varepsilon_{n-1},$$

where  $A'_j$  is also a linear combination of  $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}$  with integer coefficients, and

$j \neq n$  also. If the same coordinate  $x_i$  occurs in each of these two systems of equations, then either they are not compatible and the intersection of all  $r$  cubes is empty, or

$$A_i + a_n \varepsilon_{n-1} = A'_i + (a_n - 1) \varepsilon_{n-1},$$

and hence  $\varepsilon_{n-1} = A'_i - A_i$  is a linear combination of  $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-2}$  with integer coefficients, contrary to the assumption. The contradiction shows that if the intersection is non-empty, then it is contained in the plane defined by  $r_0 - 1 + r_1 - 1 = r - 2$  linearly independent equations. Together with  $x_n = a_n$  we get  $r - 1$  equations, as claimed. ■

**Lebesgue tiling theorem.** *For every  $\varepsilon > 0$  there exists a closed  $\varepsilon$ -covering of  $[0, 1]^n$  of order  $n + 1$ .*

**Proof.** A scaled version of the above covering of  $\mathbb{R}^n$  consists of cubes with sides of any given length. The cube  $[0, 1]^n$  is contained in the union of finitely many cubes of the scaled covering, and it is sufficient to take their intersections with  $[0, 1]^n$ . ■

**Remark.** A converse to Theorem 2.1 is also true. Namely, if for every  $r$  the intersection of  $r$  different cubes of the form  $e(a_1, a_2, \dots, a_n)$  (with integer  $a_1, a_2, \dots, a_n$ ) is contained in a plane of dimension  $n + 1 - r$ , then  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  are generic. This can be proved by inverting the arguments in the proof of Theorem 2.1. We leave details to reader.

**Intersections of cubes in Lebesgue tilings.** Now we turn to the pattern of intersections of cubes of a Lebesgue tiling. Of course, the choice of  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  is assumed to be generic. In particular, none of numbers  $\varepsilon_i$  is an integer. Clearly, replacing  $\varepsilon_i$  by  $\varepsilon_i + k_i$  with integer  $k_i$  does not change the partition. We prefer  $\varepsilon_i$  to be negative in order to match better some further constructions and will assume that  $-1 < \varepsilon_i < 0$  for every  $i \leq n - 1$ . We will also assume that  $|\varepsilon_1| + |\varepsilon_2| + \dots + |\varepsilon_{n-1}| < 1$ .

**2.2. Lemma.** *If the cubes  $e(a_1, a_2, \dots, a_n)$  and  $e(b_1, b_2, \dots, b_n)$  intersect, then*

$$(9) \quad b_i \leq a_i + 1 + (a_{i+1} - b_{i+1})\varepsilon_i + \dots + (a_n - b_n)\varepsilon_{n-1}$$

*for every  $i \leq n$  (without any assumptions about the choice of  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$ ).*

**Proof.** If these cubes intersect, then the intervals

$$[a_i + a_{i+1}\varepsilon_i + \dots + a_n\varepsilon_{n-1}, a_i + 1 + a_{i+1}\varepsilon_i + \dots + a_n\varepsilon_{n-1}]$$

and

$$[b_i + b_{i+1}\varepsilon_i + \dots + b_n\varepsilon_{n-1}, b_i + 1 + b_{i+1}\varepsilon_i + \dots + b_n\varepsilon_{n-1}]$$

overlap for every  $i \leq n$ . It follows that

$$b_i + b_{i+1}\varepsilon_i + \dots + b_n\varepsilon_{n-1} \leq a_i + 1 + a_{i+1}\varepsilon_i + \dots + a_n\varepsilon_{n-1}.$$

The last inequality obviously implies (9). ■

**2.3. Lemma.** *Under the above assumptions, if  $e(a_1, a_2, \dots, a_n)$  and  $e(b_1, b_2, \dots, b_n)$  intersect, then  $|a_i - b_i| \leq 1$  for every  $i \leq n$ .*

**Proof.** Suppose that  $i \leq n$  and  $|a_j - b_j| \leq 1$  for  $j \geq i + 1$ . Lemma 2.2 implies that

$$\begin{aligned} b_i &\leq a_i + 1 + |a_{i+1} - b_{i+1}| \cdot |\varepsilon_i| + \dots + |a_n - b_n| \cdot |\varepsilon_{n-1}|. \\ &\leq a_i + 1 + |\varepsilon_i| + \dots + |\varepsilon_{n-1}| \leq a_i + 1 + |\varepsilon_1| + \dots + |\varepsilon_{n-1}|. \end{aligned}$$

But  $|\varepsilon_1| + \dots + |\varepsilon_{n-1}| < 1$  and hence  $b_i < a_i + 2$ . Since  $a_i$  and  $b_i$  are integers, in fact  $b_i \leq a_i + 1$ . By interchanging the roles of  $a_i$  and  $b_i$ , we conclude that also  $a_i \leq b_i + 1$ . It follows that  $|a_i - b_i| \leq 1$ . It remains to use the descending induction by  $i$ . ■

**2.4. Lemma.** *Under the above assumptions, if  $e(a_1, a_2, \dots, a_n)$  and  $e(b_1, b_2, \dots, b_n)$  intersect, then either  $a_i \leq b_i$  for every  $i \leq n$ , or  $b_i \leq a_i$  for every  $i \leq n$ .*

**Proof.** Suppose that the lemma holds with  $n - 1$  in the role of  $n$ . Then it is true in the case when both cubes are contained in the same layer  $\mathcal{L}(a)$ . The cubes  $e(a_1, a_2, \dots, a_n)$  and  $e(b_1, b_2, \dots, b_n)$  are contained in the layers  $\mathcal{L}(a_n)$  and  $\mathcal{L}(b_n)$  respectively. If these layers are different, then  $|a_n - b_n| = 1$ . We may assume that  $a_n = b_n + 1$ . Suppose that  $i < n$  and  $a_j \geq b_j$  for  $j \geq i + 1$ . By Lemma 2.2

$$b_i \leq a_i + 1 + (a_{i+1} - b_{i+1})\varepsilon_i + \dots + (a_n - b_n)\varepsilon_{n-1}.$$

Since the numbers  $\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_{n-1}$  are negative,  $a_n - b_n = 1$ , and  $a_j - b_j \geq 0$  for  $j \geq i + 1$ , this inequality implies that  $b_i \leq a_i + 1 + \varepsilon_{n-1} < a_i + 1$ . Since  $a_i$  and  $b_i$  are integers, it follows that  $b_i \leq a_i$ . It remains to use the descending induction by  $i$ . ■

**Discrete cubes and a partial order on  $\mathbb{R}^n$ .** The last two lemmas suggest the following two definitions. A *discrete  $n$ -cube* is defined as a subset of  $\mathbb{Z}^n$  of the form

$$\prod_{i=1}^n \{z_i, z_i + 1\},$$

where  $z_1, z_2, \dots, z_n \in \mathbb{Z}$ . By the definition  $(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n)$  means that  $a_i \leq b_i$  for all  $i \leq n$ . As usual,  $u < v$  means that  $u \leq v$  and  $u \neq v$ .

**Pivot (sub)sequences.** Let  $v(0), v(1), \dots, v(r) \in \mathbb{Z}^n$  be pairwise  $\leq$ -comparable distinct  $n$ -tuples of integers. After reordering them, if necessary, we may assume that

$$v(0) < v(1) < \dots < v(r).$$

Since  $v(i-1) < v(i)$ , the  $n$ -tuple  $v(i)$  can be obtained from  $v(i-1)$  by increasing several coordinates. Suppose that, in addition, the  $n$ -tuples  $v(0), v(1), \dots, v(r)$  are contained in a discrete  $n$ -cube. Then each coordinate may increase only by 1 and only once along the sequence  $v(0), v(1), \dots, v(r)$ . This implies, in particular, that  $r \leq n$ . Clearly, the directions of vectors from  $v(i-1)$  to  $v(i)$  are all different. Let us define a *pivot subsequence* as an  $<$ -increasing sequence of elements of a discrete  $n$ -cube. A *pivot sequence* is a pivot subsequence consisting of  $n+1$  terms.

In a pivot sequence each coordinate increases by 1 as we go along the sequence, and only once. Increasing  $m$  coordinates at one step in a pivot subsequence can be replaced by increasing them one by one in  $m$  steps. It follows that every pivot subsequence is a subsequence of a pivot sequence, justifying the term.

**Sequences of cubes associated with pivot subsequences.** Let  $v(0) < v(1) < \dots < v(r)$  be a pivot subsequence. For  $i = 0, 1, \dots, r$  let

$$v(i) = (a_1(i), a_2(i), \dots, a_n(i)),$$

and let the real numbers

$$u_1(i), u_2(i), \dots, u_n(i)$$

be related to the integers

$$a_1(i), a_2(i), \dots, a_n(i)$$

by an obvious modification of the formulas (7). Let

$$e(i) = e(a_1(i), a_2(i), \dots, a_n(i)) = c(u_1(i), u_2(i), \dots, u_n(i)).$$

**2.5. Theorem.** *Under the above assumptions*

$$e(0) \cap e(1) \cap \dots \cap e(r) = \prod_{i=1}^n J_i,$$

where  $n-r$  factors  $J_i$  are non-degenerate closed intervals in  $\mathbb{R}$  and  $r$  factors  $J_i$  are one-point subsets of  $\mathbb{R}$ . In particular,  $e(0) \cap e(1) \cap \dots \cap e(r)$  is non-empty.

**Proof.** By the definitions

$$e(0) \cap e(1) \cap \dots \cap e(r) = \bigcap_{i=0}^r c(u_1(i), u_2(i), \dots, u_n(i)).$$

Clearly, the latter intersection is a product of  $n$  intervals, namely, of the intervals

$$U_k = \bigcap_{i=0}^r [u_k(i), u_k(i) + 1]$$

with  $k = 1, 2, \dots, n$ .

Let us temporarily fix  $k$  and write  $u(i)$  for  $u_k(i)$ . For an integer  $i$  between 1 and  $r$  let  $\sigma(i)$  be the sum of the numbers  $\varepsilon_j$  such that  $j \geq k + 1$  and

$$a_j(i) = a_j(i - 1) + 1.$$

The sums  $\sigma(i)$  are negative and since  $v(i)$  is a pivot subsequence, different sums  $\sigma(i)$  involve different numbers  $\varepsilon_j$ . Together with our assumptions about  $\varepsilon_j$  this implies that

$$\sigma(s + 1) + \sigma(s + 2) + \dots + \sigma(t) + 1 > 0.$$

if  $0 \leq s < t \leq r$ . Suppose now that

$$a_k(s) = a_k(s + 1) = \dots = a_k(t).$$

Then the interval

$$[u(i), u(i) + 1]$$

with  $s + 1 \leq i \leq t$  is obtained by shifting the interval

$$[u(i - 1), u(i - 1) + 1]$$

by the amount  $|\sigma(i)|$  to the left. The total amount of shifting

$$\text{from } [u(s), u(s) + 1] \text{ to } [u(t), u(t) + 1]$$

is equal to  $|\sigma(s + 1) + \sigma(s + 2) + \dots + \sigma(t)| < 1$ . It follows that

$$(10) \quad \bigcap_{i=s}^t [u(i), u(i) + 1] = [u(s), u(t) + 1]$$

and this interval is non-empty and, moreover, proper.

In particular, if

$$a_k(0) = a_k(1) = \dots = a_k(r),$$

then  $U_k$  is equal to the proper interval  $[u(r), u(0) + 1]$ .

Suppose now that not all values  $a_k(i)$  are equal. Then there is a unique  $m$  such that

$$a_k(0) = a_k(1) = \dots = a_k(m-1),$$

$$a_k(m) = a_k(m-1) + 1, \quad \text{and}$$

$$a_k(m) = a_k(m+1) = \dots = a_k(r).$$

Clearly,

$$\begin{aligned} u(m) &= u(m-1) + \sigma(m) + 1 \\ &= u(0) + \sigma(1) + \dots + \sigma(m-1) + \sigma(m) + 1. \end{aligned}$$

It follows that

$$u(0) < u(m) \leq u(m-1) + 1$$

and  $u(m) = u(m-1) + 1$  if and only if  $\sigma(m) = 0$ . Also,

$$\begin{aligned} u(r) &= u(m) + \sigma(m+1) + \dots + \sigma(r) \\ &= u(m-1) + 1 + \sigma(m) + \sigma(m+1) + \dots + \sigma(r) > u(m-1). \end{aligned}$$

By applying (10) with  $s = 0$  and  $t = m-1$  we see that

$$\bigcap_{i=0}^{m-1} [u(i), u(i) + 1] = [u(0), u(m-1) + 1].$$

Similarly, by applying (10) with  $s = m$  and  $t = r$  we see that

$$\bigcap_{i=m}^r [u(i), u(i) + 1] = [u(m), u(r) + 1]$$

By combining the last two displayed formulas conclude that

$$U_k = \bigcap_{i=0}^r [u(i), u(i) + 1] = [u(0), u(m-1) + 1] \cap [u(m), u(r) + 1].$$



The inequalities  $u(0) < u(m) \leq u(m-1) + 1 < u(r) + 1$ , proved above, imply that

$$[u(0), u(m-1) + 1] \cap [u(m), u(r) + 1] = [u(m), u(m-1) + 1]$$

and hence  $U_k = [u(m), u(m-1) + 1]$ . Since  $u(m) \leq u(m-1)$ , this interval is non-empty. It is proper if and only if  $u(m) < u(m-1) + 1$ , or, equivalently,  $\sigma(m) < 0$ . The latter condition is equivalent to  $a_j(m+1) = a_j(m) + 1$  for some  $j \geq k+1$ . Equivalently, the interval  $U_k$  is proper if and only if  $a_k(i)$  increases at the same step as some  $a_j(i)$  with  $j > k$ .

Since  $k$  was arbitrary, this proves that the intersection  $e(0) \cap e(1) \cap \dots \cap e(r)$  is non-empty and is a product of closed intervals. It remains to determine the number of proper intervals among them. Let us return to treating  $k$  as an arbitrary integer between 1 and  $n$ . As we saw, if  $a_k(i)$  is independent of  $i$ , then the interval  $U_k$  is proper. Also, let

$$a_{k_1}(i), a_{k_2}(i), \dots, a_{k_N}(i)$$

be the coordinates increasing when  $i$  changes from  $m-1$  to  $m$ . If  $N > 1$ , then

$$U_{k_1}, U_{k_2}(i), \dots, U_{k_{N-1}}(i)$$

are proper intervals. There are no other proper intervals. Let us look how the sum

$$\Sigma(i) = a_1(i) + a_2(i) + \dots + a_n(i)$$

changes when  $i$  changes from 0 to  $r$ . Each  $a_k(i)$  may stay constant or increase by 1 and hence  $\Sigma(r) = \Sigma(0) + n - c$ , where  $c$  is the number of coordinates staying constant. At the same time  $\Sigma(m) = \Sigma(m-1) + N_m$ , where  $N_m$  is the number of coordinates increasing from  $m-1$  to  $m$ . It follows that  $n - c = N_1 + N_2 + \dots + N_r$  and hence

$$n - r = (N_1 - 1) + (N_2 - 1) + \dots + (N_r - 1) + c.$$

But the right hand of this equality is equal to the number of proper intervals in the product. It follows that this number is equal to  $n - r$ . This completes the proof. ■

**2.6. Corollary.** *Under the above assumptions, the intersection of several cubes of the form  $e(a_1, a_2, \dots, a_n)$  is non-empty if and only if the corresponding  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  are the terms of a pivot subsequence. In particular, the intersection of  $n+2$  distinct cubes of the form  $e(a_1, a_2, \dots, a_n)$  is empty.* ■

**Remark.** Theorem 2.5 and Corollary 2.6 provide another proof of Lebesgue tiling theorem.

### 3. Modification of Lebesgue methods by W. Hurewicz

**Tilings.** The intersections of  $n$ -cubes of  $Q$  are rather excessive, if compared with Lebesgue covering theorems: vertices of  $n$ -cubes, except of ones belonging to the boundary  $\text{bd } Q$ , are contained in  $2^n$  different  $n$ -cubes. At the same time, Lebesgue tilings consist of similar cubes with no more than  $n + 1$  intersecting. W. Hurewicz [H2] suggested to start with a partition of a cube or of a product of intervals already having this property.

Following W. Hurewicz, let us call  $n$ -intervals the products of the form

$$Z = \prod_{i=1}^n J_i \subset \mathbb{R}^n,$$

where  $J_i = [a_i, b_i]$  for some  $a_i < b_i$  for every  $i = 1, 2, \dots, n$ . More generally, an  $m$ -interval is a product of the same form, but with  $m$  factors  $J_i$  being intervals  $[a_i, b_i]$  with  $a_i < b_i$  as before and the other  $n - m$  factors being one-point subsets of  $\mathbb{R}$ . By an *interval* we will understand an  $m$ -interval for some  $m$ . The usual intervals will be called *intervals in  $\mathbb{R}$* . The *faces* and the *boundary*  $\text{bd } Z$  of an interval  $Z$  are defined in the obvious manner.

For the rest of this section we will assume that an  $n$ -interval  $Z$  as above is fixed. We will denote by  $A_i$  and  $B_i$  the  $(n - 1)$ -faces of  $Z$  obtained by replacing in the above product the factor  $[a_i, b_i]$  by  $\{a_i\}$  and  $\{b_i\}$  respectively. A *partition* of  $Z$  is a finite collection of  $n$ -intervals such that  $Z$  is equal to their union and their interiors are pair-wise disjoint. A partition of  $Z$  is called a *tiling* if its intervals satisfy the following two conditions:

- (a) No point belongs to more than  $n + 1$  of intervals.
- (b) The intersection of  $k$  intervals is either empty, or is an  $(n + 1 - k)$ -interval.

From now on we will denote by  $Z_1, Z_2, \dots, Z_m$  a tiling of  $Z$  and call its elements  $Z_i$  *tiles*.

**Tilings into small  $n$ -intervals.** Of course, one needs to know that tilings consisting of arbitrarily small  $n$ -intervals exist. Theorem 2.5 and Corollary 2.6 imply that Lebesgue tilings satisfy conditions (a) and (b). By scaling and intersecting the scaled cubes with  $Z$  one can get a tiling of  $Z$  by arbitrary small tiles. It seems that Hurewicz wasn't interested in the detailed description of intersections in Lebesgue tilings provided by Theorem 2.5 (and its proof), but at his disposal was Theorem 2.1, due to Lebesgue [L2]. Hurewicz [H2] mentioned in a footnote that if no interval of a partition intersects two opposite  $(n - 1)$ -faces of  $Z$ , then (a) implies (b). While this fact together with Lebesgue results (i.e. together with Theorem 2.1) also implies that Lebesgue tilings satisfy conditions (a) and (b), Hurewicz did not write down a proof. Instead, he outlined (in another footnote) a more simple and general way to construct the needed tilings. See the next theorem.

**3.1. Theorem.** *There exist tilings of  $Z$  consisting of arbitrarily small  $n$ -intervals.*

**Hurewicz's proof.** Let  $(x_1, x_2, \dots, x_n)$  be the standard coordinates in  $\mathbb{R}^n$ . To begin with, the partition consisting only of  $Z$  is trivially a tiling. Let  $Z_1, Z_2, \dots, Z_m$  be a tiling of  $Z$ . Let  $Z_k$  be an  $n$ -interval of this tiling, and let us divide  $Z_k$  into two  $n$ -intervals

$$Z'_k, Z''_k$$

by a hyperplane  $H \subset \mathbb{R}^n$  defined by an equation of the form  $x_i = a$ , where  $a \in \mathbb{R}$ . If the hyperplane  $H$  does not contain any  $(n-1)$ -face of any tile  $Z_1, Z_2, \dots, Z_m$ , then replacing  $Z_k$  by the pair  $Z'_k, Z''_k$  results in a tiling. By repeating this procedure with hyperplanes of various directions one can get tilings consisting of arbitrarily small intervals. ■

**3.2. Lemma.** *Let  $B$  be an  $(n-1)$ -face of  $Z$ . The non-empty sets among the intersections  $Z_i \cap B$ , where  $i = 1, 2, \dots, m$ , form a tiling of  $B$ .*

**Proof.** If  $x \in Z_i \cap B$ , then  $Z_i$  contains a 1-interval having  $x$  as one of its endpoints and orthogonal to  $B$ . It follows that taking the intersection with  $B$  decreases the dimension of any interval of the form  $Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_s}$  by 1. In turn, this implies the lemma. ■

**3.3. Lemma.** *Let  $z$  be the intersection of  $n$  tiles. If  $z \neq \emptyset$ , then  $z$  is a 1-interval such that each of its endpoints either belongs to an  $(n-1)$ -face of  $Z$ , or is contained in  $n+1$  tiles. Other points of  $z$  are contained in only  $n$  tiles.*

**Proof.** Without any loss of generality we can assume that  $z = Z_1 \cap Z_2 \cap \dots \cap Z_n$ . Suppose that  $p$  is an endpoint of  $z$  not contained in an  $(n-1)$ -face of  $Z$ . Let  $H$  be a hyperplane in  $\mathbb{R}^n$  containing  $p$  and orthogonal to  $z$ , and let us extend  $z$  to a 1-interval  $z'$  containing  $p$  in its interior. Every point of the difference  $z' \setminus z$  is not contained in tiles  $Z_i$  with  $i \leq n$ . It follows that for some  $j \leq n$  all points of  $z' \setminus z$  sufficiently close to  $p$  do not belong to  $Z_j$ . Since  $p \in Z_j$ , the intersection  $Z_j \cap H$  is an  $(n-1)$ -face of  $Z_j$ .

Since the tiles form a partition, there is a tile  $Z_k$  different from  $Z_j$  and such that  $Z_k \cap H$  is an  $(n-1)$ -face of  $Z_k$ . Clearly,  $Z_k$  and  $Z_j$  cannot be at the same side of  $H$ . It follows that  $Z_k$  is different from all  $Z_i$  with  $i \leq n$  and hence  $p$  is contained in  $n+1$  tiles.

Suppose now that  $p \in Z_k$  for some  $k > n$ . Then  $p$  is the only point of the intersection  $Z_1 \cap Z_2 \cap \dots \cap Z_n \cap Z_k = z \cap Z_k$ . But the intersection of a 1-interval and an  $n$ -interval may consist of only one point only if this point is an endpoint of the 1-interval. ■

**Tiled subsets of  $Z$ .** A *tiled* subset of  $Z$  is defined as the union of several tiles. Tiled subsets of  $Z$  are the analogues of cubical subsets of  $Q$  from Section 1. Two tiled subsets are said to be *essentially disjoint* if they are unions of two disjoint collections of tiles.

**3.4. Lemma.** Suppose that  $e_1, e_2, \dots, e_{n+1}$  is a covering of  $Z$  by  $n + 1$  tiled subsets, and that these tiled subsets are pairwise essentially disjoint. Let

$$P = e_1 \cap e_2 \cap \dots \cap e_n.$$

Then  $P$  is the union of several 1-intervals  $z_1, z_2, \dots, z_f$  intersecting only at their endpoints and such that each endpoint of each interval  $z_i$  is the endpoint of 1 or 2 of these intervals. If  $p$  is the endpoint of only one interval and  $p \notin \text{bd } Z$ , then  $p \in e_1 \cap e_2 \cap \dots \cap e_{n+1}$ .

**Proof.** Let us consider intersections of the form  $Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_n}$  with  $Z_{i_k} \subset e_k$  for every  $k \leq n$ . Let  $z_1, z_2, \dots, z_f$  be the non-empty sets among these intersections. Since  $E_i$  are tiled sets,  $P$  is equal to the union  $z_1 \cup z_2 \cup \dots \cup z_f$ . Since  $e_1, e_2, \dots, e_{n+1}$  are pairwise essentially disjoint, the property (b) of tilings implies that each  $z_i$  is a 1-interval. Let  $p$  be an endpoint of some such 1-interval

$$z_t = Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_n}.$$

Suppose that  $p \in \text{bd } Z$ . Lemma 3.2 implies that  $\text{bd } Z$  does not contain any of the intervals  $z_i$ . It follows that in this case  $p$  is the endpoint only of  $z_t$ .

Suppose now that  $p \notin \text{bd } Z$ . Then Lemma 3.3 implies that there is a tile  $Z_{i_{n+1}}$  containing  $p$  and not equal to any  $Z_{i_k}$  with  $k \leq n$ . By the property (b) of tilings, the intersection

$$Z_{i_1} \cap Z_{i_2} \cap \dots \cap Z_{i_n} \cap Z_{i_{n+1}}$$

is a 0-interval and hence contains only one point, namely  $p$ . The property (a) of tilings implies that  $p$  does not belong to any tiles other than  $Z_{i_k}$  with  $k \leq n + 1$ .

There are two cases to consider. First, suppose that  $p \in e_{n+1}$ . Since  $e_{n+1}$  is essentially disjoint from each  $e_i$  with  $i \leq n$ , in this case  $Z_{i_{n+1}} \subset e_{n+1}$ . In view of the observation at the end of the previous paragraph, this implies that  $p$  is not an endpoint of any 1-interval  $z_i$  different from  $z_t$ . In other words, in this case  $p$  is also the endpoint only of  $z_t$ .

If  $p \notin e_{n+1}$ , then  $Z_{i_{n+1}} \subset e_k$  for some  $k \leq n$  and among the  $n + 1$  tiles

$$Z_{i_1}, Z_{i_2}, \dots, Z_{i_n}, Z_{i_{n+1}}$$

there are exactly 2 sets of  $n$  tiles including one tile from each of  $e_1, e_2, \dots, e_n$ , namely

$$Z_{i_1}, Z_{i_2}, \dots, Z_{i_n} \quad \text{and} \quad Z_{i_1}, \dots, Z_{i_{k-1}}, Z_{i_{k+1}}, \dots, Z_{i_{n+1}}.$$

The intersection of tiles from the first set is  $z_t$  and the intersection of tiles from the second set is some other  $z_u$ . Clearly,  $p$  is the endpoint of  $z_t$  and  $z_u$ , but of no other interval  $z_i$ . In this case  $p$  is the endpoint of exactly 2 intervals  $z_i$ . The lemma follows. ■

**3.5. Theorem.** *Let  $e_1, e_2, \dots, e_{n+1}$  be as in Lemma 3.4. Suppose that  $e_i$  is disjoint from  $B_i$  for every  $i \leq n$  and is disjoint from  $A_{i-1}$  for every  $i \geq 2$ . Then the intersection of the sets  $e_1, e_2, \dots, e_{n+1}$  consists of an odd number of points.*

**Proof.** Let us use the induction by  $n$ . Suppose that  $n = 1$ . Then  $Z$  is a 1-interval partitioned into several 1-intervals with disjoint interiors. There are two sets,  $e_1$  and  $e_2$  which are unions of two disjoint collections of these intervals and such that  $Z = e_1 \cup e_2$ . It is assumed that one endpoint of  $Z$  belongs to  $e_1$  and the other to  $e_2$ . Therefore, if one moves along  $Z$  from one endpoint to the other, one has to pass from  $e_1$  to  $e_2$  or vice versa an odd number of times. This means that  $e_1 \cap e_2$  consists of an odd number of points.

Suppose now that  $n > 1$  and that the theorem is already proved for  $n - 1$  in the role of  $n$ . By Lemma 3.2 the non-empty sets among the intersections  $Z_i \cap A_n$  form a tiling of the  $(n - 1)$ -face  $A_n$  of  $Z$ . Since  $e_{n+1}$  is disjoint from  $A_n$ , the sets

$$e_1 \cap A_n, e_2 \cap A_n, \dots, e_n \cap A_n$$

cover  $A_n$ . Obviously, they are tiled subsets of  $A_n$  and are essentially disjoint. By the inductive assumption the intersection  $P \cap A_n$  of these sets consists of an odd number of points.

At the same time the assumptions of the theorem imply that  $P$  is disjoint from  $B_i$  for all  $i \leq n$  and is disjoint from  $A_i$  for all  $i \leq n - 1$ . Therefore  $P \cap \text{bd} Z = P \cap A_n$ .

Let  $z_1, z_2, \dots, z_f$  be the intervals from Lemma 3.4. Let  $e$  and  $h$  be the numbers of endpoints of these intervals belonging to  $e_1 \cap e_2 \cap \dots \cap e_{n+1}$  and  $A_n$  respectively. By the previous paragraph  $h$  is also equal to the number of endpoints of these intervals belonging to the boundary  $\text{bd} Z$ . Let  $r$  be the number of endpoints in the interior  $Z \setminus \text{bd} Z$  of  $Z$  which are endpoints of exactly two of the intervals. Now Lemma 3.4 implies that

$$e + h + 2r = 2f.$$

Since  $h$  is odd by the inductive assumption, it follows that  $e$  is also odd. Since the sets  $e_i$  are essentially disjoint, every point of the intersection  $e_1 \cap e_2 \cap \dots \cap e_{n+1}$  is contained in  $n + 1$  different tiles, one from each  $e_i$ . In view of Lemma 3.3 this implies that every point of this intersection is an endpoint of one of the 1-intervals  $z_i$ . It follows that  $e$  is the number of points of this intersection, and hence  $e_1 \cap e_2 \cap \dots \cap e_{n+1}$  consists of an odd number of points. This completes the step of the induction and hence the proof. ■

**3.6. Theorem.** *Let  $d_1, d_2, \dots, d_r$  be a covering of  $Z$  by tiled sets. Suppose that none of the sets  $d_i$  intersects two opposite  $(n - 1)$ -faces of  $Z$ . Then among the sets  $d_i$  there are  $n + 1$  sets with non-empty intersection.*

**Proof.** It is sufficient to combine Theorem 3.5 with Lebesgue fusion construction. ■

**Remark.** At the corresponding place of his paper [H2] Hurewicz wrote that

One should point out emphatically the combinatorial nature of the preceding considerations. The apparent use of the geometry of continuous figures served only the goal of simplifying the exposition.

See [H2], footnote 28 on p. 217. The combinatorial nature of Hurewicz's proofs manifests itself best if they are translated into the simplicial language. See Section 4.

**Lebesgue first covering theorem.** *Let  $D_1, D_2, \dots, D_r$  be a covering of the unit cube  $[0, 1]^n$  by closed sets. Suppose that none of the sets  $D_i$  intersects two opposite  $(n-1)$ -faces of  $[0, 1]^n$ . Then among the sets  $D_i$  there are  $n+1$  sets with non-empty intersection.*

**Hurewicz's proof.** The main part of work is already done. Given Theorem 3.6, the rest of proof is almost the same as the proof in Section 1. Let  $Z = [0, 1]^n$  and let us choose a tiling of  $Z$  consisting of tiles of diameter  $< \varepsilon$ , where  $\varepsilon$  is the same as in Section 1. For each  $i \leq r$  let  $d_i$  be the union of all tiles intersecting  $D_i$ . Now one can use Theorem 3.6 instead of Theorem 1.6. We leave details to the reader. ■

**3.7. Theorem.** *Let  $d_1, d_2, \dots, d_{n+1}$  be a covering of  $Z$  by  $n+1$  tiled subsets. Suppose that  $d_i$  is disjoint from  $B_i$  for every  $i \leq n$  and is disjoint from  $A_{i-1}$  for every  $i \geq 2$ . Then the intersection of the sets  $d_1, d_2, \dots, d_{n+1}$  is non-empty.*

**Proof.** It is based on the same idea as the proof of Theorem 1.7. For  $i \leq n+1$  let  $e_i$  be the union of all tiles contained in  $d_i$ , but not in any  $d_j$  with  $j < i$ . Then

$$e_1 \cup e_2 \cup \dots \cup e_i = d_1 \cup d_2 \cup \dots \cup d_i$$

for every  $i$ . In particular,  $e_1, e_2, \dots, e_{n+1}$  is a covering of  $Z$  by  $n+1$  tiled subsets. By the construction, these sets are essentially disjoint. On the other hand,  $e_i \subset d_i$  and hence  $e_i$  is disjoint from  $B_i$  for every  $i \leq n$  and is disjoint from  $A_{i-1}$  for every  $i \geq 2$ . It follows that the sets  $e_1, e_2, \dots, e_{n+1}$  satisfy the assumptions of Theorem 3.5 and hence  $e_1 \cap e_2 \cap \dots \cap e_{n+1} \subset d_1 \cap d_2 \cap \dots \cap d_{n+1}$  is non-empty. ■

**3.8. Theorem.** *Let  $D_1, D_2, \dots, D_{n+1}$  be a covering of the unit cube  $[0, 1]^n$  by closed sets. Suppose that  $D_i$  is disjoint from  $B_i$  for every  $i \leq n$  and is disjoint from  $A_{i-1}$  for every  $i \geq 2$ . Then the intersection of the sets  $D_1, D_2, \dots, D_{n+1}$  is non-empty.*

**Proof.** It is completely similar to Hurewicz's proof of Lebesgue first covering theorem, with Theorem 3.7 playing the role of Theorem 3.6. ■

**Remark.** This is a stronger version of Theorem 1.9.

## 4. Nerves of coverings and Hurewicz's theorems

**The nerve of a finite system of sets.** Let

$$M_1, M_2, \dots, M_r$$

be a system of sets. In the present context such system usually arises as a covering of a set under consideration. The *nerve* of this system is an abstract simplicial complex constructed as follows. Its vertices  $v_1, v_2, \dots, v_r$  are in a one-to-one correspondence with the sets of the system. A set  $\{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$  of vertices is a simplex if and only if

$$M_{i_1} \cap M_{i_2} \cap \dots \cap M_{i_s} \neq \emptyset.$$

This notion was introduced in 1926 by P. Alexandroff [A-f]. He had in mind the goal of relating general topological spaces with simplicial complexes, either abstract ones, which are pure combinatorial finite objects, or geometric ones, which are geometric figures of finite nature. The notion of a nerve turned out to be extremely fruitful far beyond this initial (already very ambitious) goal, but we will use it only as a convenient language.

Probably, the simplest example is provided by the collection of  $(n-1)$ -faces of a geometric  $n$ -simplex  $\Delta$ . It is a covering of the boundary  $\text{bd } \Delta$ . The intersection of all  $(n-1)$ -faces is empty, but the intersection of any proper subset of the set of  $(n-1)$ -faces is not. Therefore, the nerve of this covering of the boundary  $\text{bd } \Delta$  is the boundary of an abstract  $n$ -simplex (considered as a pseudo-manifold). The reader may try to identify the nerve of the covering of the boundary of an  $n$ -cube by its  $(n-1)$ -faces. As a more general example, let us consider a geometric simplicial complex  $S$  and the covering of its polyhedron  $\|S\|$  by the closed barycentric stars. The discussion in Section 3 of [I] implies that the nerve of this covering is isomorphic to the corresponding abstract simplicial complex.

**Infinite systems of sets.** The notion of the nerve easily extends to infinite systems of sets, once the notion of an abstract simplicial complex is extended to allow infinite sets of vertices. This is also easy: the sets of vertices and simplices are allowed to be infinite, but the simplices are still required to be finite. So, a potentially infinite abstract simplicial complex  $K$  is defined as a collection  $K$  of *finite* subsets of a set  $v(K)$  such that if  $\sigma \in K$  and  $\sigma' \subset \sigma$ , then  $\sigma' \in K$ , and  $v(K)$  is equal to the union of all subsets in  $K$ . The elements of  $v(K)$  are called *vertices*, and the elements of  $K$  the *simplices* of  $K$ .

Now the nerve of an arbitrary system  $\mathfrak{M}$  of sets is defined in the same way as before. Its vertices  $v_m$  are in a one-to-one correspondence with the sets of the system, i.e. with the elements  $m \in \mathfrak{M}$ . A set  $\{v_{m_1}, v_{m_2}, \dots, v_{m_s}\}$  of vertices is a simplex if and only if

$$m_1 \cap m_2 \cap \dots \cap m_s \neq \emptyset.$$

**The nerve of Lebesgue tilings.** We are going to discuss essentially only one infinite system of sets, namely, Lebesgue tilings of  $\mathbb{R}^n$ . We keep the notations and assumptions of Section 2. In particular, we will assume that the numbers  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  are generic, such that  $-1 < \varepsilon_i < 0$  for every  $i \leq n-1$ , and such that  $|\varepsilon_1| + |\varepsilon_2| + \dots + |\varepsilon_{n-1}| < 1$ .

The cubes  $e(a_1, a_2, \dots, a_n)$  of a Lebesgue tiling are in a canonical one-to-one correspondence with elements of  $\mathbb{Z}^n$ . Therefore, the nerve of a Lebesgue tiling may be considered as an infinite abstract simplicial complex with  $\mathbb{Z}^n$  being the set of vertices. Corollary 2.6 provides a complete combinatorial description of this nerve. Namely, a subset of  $\mathbb{Z}^n$  is a simplex if and only if it is the set of terms of a pivot subsequence (as defined in Section 2). In particular, under the above assumptions about the parameters  $\varepsilon_i$  the nerve is independent on the choice of these parameters.

**The nerve of a tiling.** As in Section 3, let  $Z$  be an  $n$ -interval and let  $Z_1, Z_2, \dots, Z_m$  be a tiling of  $Z$ . We will assume that no tile  $Z_i$  intersects two opposite faces of  $Z$ . Let  $\mathcal{Z}$  be the nerve of the covering of  $Z$  by the tiles, and let  $\mathcal{V} = \{v_1, v_2, \dots, v_m\}$  be the set of its vertices, the vertex  $v_i$  corresponding to the tile  $Z_i$ . By the definition of tilings, the dimension of  $\mathcal{Z}$  is  $\leq n$  (Lebesgue covering theorems imply that it is equal to  $n$ ).

For each integer  $i$  between 1 and  $n$  let  $\mathcal{A}_i$  be the nerve of the system of sets consisting of tiles intersecting  $A_i$ . Let us define  $\mathcal{B}_i$  in the same manner, and let  $\text{boundary } \text{bd } \mathcal{Z}$  be the union of simplicial complexes  $\mathcal{A}_i, \mathcal{B}_i$  over all  $1 \leq i \leq n$ . The assumption that no tile intersects two opposite faces of  $Z$  implies that  $\mathcal{A}_i$  is disjoint from  $\mathcal{B}_i$  for every  $i$ . Lemma 3.2 implies that the dimension of  $\text{bd } \mathcal{Z}$  is  $\leq n-1$  (in fact, it is equal to  $n-1$ ).

By the definition of the simplicial complex  $\mathcal{Z}$ , its  $n$ -simplices correspond to non-empty intersections of  $n+1$  tiles, and its  $(n-1)$ -simplices correspond to non-empty intersections of  $n$  tiles. The latter are exactly the 1-intervals considered in Lemma 3.3. Since we assumed that no tiles intersect both  $A_i$  and  $B_i$ , these 1-intervals cannot have endpoints in both  $A_i$  and  $B_i$ . If such an 1-interval has an endpoint in  $A_i$  or  $B_i$ , then the corresponding  $(n-1)$ -simplex is a simplex of  $\mathcal{A}_i$  or  $\mathcal{B}_i$  respectively.

Therefore, in the language of  $\mathcal{Z}$ , Lemma 3.3 means that an  $(n-1)$ -simplex of  $\mathcal{Z}$  is either contained in the boundary  $\text{bd } \mathcal{Z}$  and then it is a face of one  $n$ -simplex, or it is a face of two  $n$ -simplices. In other words, this lemma establishes an analogue of the non-branching property of triangulations of a geometric simplex (see [I], Section 2).

**The combinatorics of Hurewicz's method.** Let  $e_1, e_2, \dots, e_{n+1}$  be a covering of the set  $\mathcal{V}$  of vertices of  $\mathcal{Z}$  by  $n+1$  pairwise disjoint subsets. Such a covering can be interpreted as a labeling of  $\mathcal{V}$ , where the label of the vertex  $v_i$  is the unique number  $k$  such that  $v_i \in e_k$ . Even better, this covering can be interpreted as a simplicial map

$$\varphi: \mathcal{Z} \longrightarrow \Delta,$$



where  $\Delta$  is the simplicial complex having  $\{1, 2, \dots, n+1\}$  as the set of vertices and all sets of vertices as simplices (i.e. the simplex  $\{1, 2, \dots, n+1\}$  considered as a simplicial complex). It is worth to point out that these interpretations depend on the disjointness assumption.

Let  $\Delta_{n+1}$  be the  $(n-1)$ -face  $\{1, 2, \dots, n\}$  of  $\Delta$ . Then the 1-intervals  $z_k$  of Lemma 3.4 correspond to  $(n-1)$ -simplices  $\tau$  of  $\mathcal{Z}$  such that  $\varphi(\tau) = \Delta_{n+1}$ . In the language of  $\mathcal{Z}$  Lemma 3.4 means that if an  $n$ -simplex  $\sigma$  has an  $(n-1)$ -face  $\tau$  such that  $\varphi(\tau) = \Delta_{n+1}$ , then either  $\sigma$  has two such faces, or  $\varphi(\sigma) = \Delta$ .

Now everything is ready to carry out a counting argument in the spirit of Sperner's one, and this is done in the proof of Theorem 3.5. Let  $e, h, r$  and  $f$  be the numbers from this proof, and let  $g$  be the number of endpoints of the intervals  $z_i$  contained in the interior of  $Z$ . Then  $r = g - e$  and the equality

$$e + h + 2r = 2f$$

from this proof can be rewritten as  $e + h + 2(g - e) = 2f$ , or, what is the same, as

$$h + 2g = e + 2f.$$

This is the same equality as in Sperner's proof of his lemma. The Hurewicz's and Sperner's proofs differ in the context and the language, but on a deeper level they are nearly the same.

**The methods of Hurewicz and Sperner in 1928.** W. Hurewicz himself was well aware of the analogy between his methods and Sperner's ones. He wrote

During the preparation of the page proofs of this paper, an article of E. Sperner appeared (Abhandlungen des Hamburgischen Math. Sem. 6, pp. 265-472, submitted in June 1928), in which one finds a proof of Lebesgue–Brouwer theorems exhibiting a far-reaching analogy with the proof presented here.

See [H2], footnote 13a on p. 211. Hurewicz paper was submitted in January 1928, several months earlier than Sperner's paper, but was published at least several months later. In the meantime K. Menger included both proofs in his monograph [Me]. K. Menger wrote

The first proof is based on Sperner's work (already appeared in Hamburger Abhandl.) and deals with  $n$ -dimensional simplices. It simplifies Brouwer's proof by using Brouwer's notion of simplicial decompositions (Math. Ann. 71, 1911, p. 161) and suppressing the notion of degrees of maps. The second proof is Hurewicz's simplification (to appear shortly in Math. Ann.) of Lebesgue proof, dealing with  $n$ -dimensional cubes and using Lebesgue canonical cubical decompositions.

In 1928 the two methods were standing on equal footing, but their further fates are different. Hurewicz's beautiful proof wasn't included even in his book [HW] with H. Wallman.

**Lebesgue tilings and Hurewicz's theorems.** It is only natural to apply Hurewicz's theorems to Lebesgue tilings. This leads to purely combinatorial Theorems 4.1 and 4.2 below. Apparently, those who knew Hurewicz's work were not interested in such combinatorial results, and those who were interested in them, such as H.W. Kuhn [Ku], were not aware of his work.

Let  $Z$  be an  $n$ -interval. The intersections of cubes of a Lebesgue tiling with  $Z$  are  $m$ -intervals with  $m \leq n$ , and the intersections which are  $n$ -intervals form a tiling of  $Z$ . Let  $k$  be a natural number and  $K = \{0, 1, \dots, k\}^n$ . We would like to choose an  $n$ -interval  $Z$  and a Lebesgue tiling in such a way that the cubes  $e(a_1, a_2, \dots, a_n)$  with  $(a_1, a_2, \dots, a_n) \in K$  would be the ones which are intersecting  $Z$  in  $n$ -intervals. A natural way to do this is the following. Let  $\varepsilon$  be a small positive number. Assuming that  $\varepsilon < 1$  is sufficient. Let

$$Z = [\varepsilon, k + \varepsilon]^n.$$

Let  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1} \in \mathbb{R}$  be such that  $-1 < \varepsilon_i < 0$  for every  $i$  and

$$k|\varepsilon_1| + k|\varepsilon_2| + \dots + k|\varepsilon_{n-1}| < 1 - \varepsilon.$$

In particular,  $|\varepsilon_1| + |\varepsilon_2| + \dots + |\varepsilon_{n-1}| < 1$  and hence  $\varepsilon_i$  satisfy all assumptions of Section 2. Let us consider the Lebesgue tiling with the parameters  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$  and its cubes  $e(a_1, a_2, \dots, a_n)$ . A trivial check shows that  $Z$  is contained in the union of cubes  $e(a_1, a_2, \dots, a_n)$  with  $(a_1, a_2, \dots, a_n) \in K$  and such cubes intersect  $Z$  in  $n$ -intervals.

By Theorem 2.5 and Corollary 2.6 these intersections form a tiling of  $Z$ . This tiling is a part of the Lebesgue tiling of  $\mathbb{R}^n$  with the same parameters. This allows to identify the nerve  $\mathcal{K}$  of this tiling. Namely, a subset of  $K$  is a simplex of  $\mathcal{K}$  if and only if it is the set of terms of a pivot subsequence. The complexes  $\mathcal{A}_i$  and  $\mathcal{B}_i$  are also easy to identify. Namely,  $(a_1, a_2, \dots, a_n) \in K$  is a vertex of  $\mathcal{A}_i$  if and only if  $a_i = 0$ , and is a vertex of  $\mathcal{B}_i$  if and only if  $a_i = k$ . The simplices are sets of terms of pivot subsequences.

**4.1. Theorem.** *Suppose that  $e_1, e_2, \dots, e_{n+1}$  are subsets of  $K$  such that their union is equal to  $K$  and they are pairwise disjoint. Suppose that  $e_i$  is disjoint from (the set of vertices of)  $\mathcal{B}_i$  for every  $i \leq n$  and from  $\mathcal{A}_{i-1}$  for every  $i \geq 2$ . Then the number of pivot sequences having a term belonging to  $e_i$  for every  $i = 1, 2, \dots, n+1$  is odd.*

**4.2. Theorem.** *Let  $d_1, d_2, \dots, d_{n+1}$  be a covering of  $K$ . Suppose that  $d_i$  is disjoint from  $\mathcal{B}_i$  for every  $i \leq n$  and is disjoint from  $\mathcal{A}_{i-1}$  for every  $i \geq 2$ . Then there is a pivot sequence having a term belonging to  $d_i$  for every  $i = 1, 2, \dots, n+1$ .*

**Proofs.** It is sufficient to apply Theorems 3.5 and 3.7. The resulting proofs involve geometry, but mostly to simplify the exposition. The main input of geometry is the non-branching property of  $\mathcal{K}$ . See Lemma 9.1 for a combinatorial proof. ■

## 5. Lusternik–Schnirelmann theorems

**The big sphere  $S^n$ .** Let us fix a natural number  $n$ , and let  $I = \{1, 2, \dots, n+1\}$ . In this section we will work in  $\mathbb{R}^{n+1}$ . Let us also fix another natural number  $l$ , and let

$$Q = [-l - 1/2, l + 1/2]^{n+1} \subset \mathbb{R}^{n+1}.$$

Obviously,  $Q$  is invariant under the *antipodal involution*  $\iota: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$  defined by

$$\iota: x \longmapsto -x.$$

For  $X \subset \mathbb{R}^{n+1}$  we will abbreviate  $\iota(X)$  to  $\bar{X}$ . The reason for using a half-integer  $l + 1/2$  instead of an integer will be clear later. Let  $\text{pr}_i: Q \longrightarrow [-l - 1/2, l + 1/2]$  be the projection

$$(x_1, x_2, \dots, x_{n+1}) \longmapsto x_i.$$

A *face* of  $Q$  is a product of the form

$$\prod_{i=1}^{n+1} J_i,$$

where each  $J_i$  is either the interval  $[-l - 1/2, l + 1/2]$ , or the one-point set  $\{-l - 1/2\}$ , or  $\{l + 1/2\}$ . The *dimension* of a face of  $Q$  is the number of intervals in the corresponding product. A face of dimension  $m$  is also called an *m-face*. For  $i \in I$  let

$$A_i = \text{pr}_i^{-1}(-l - 1/2) \quad \text{and} \quad B_i = \text{pr}_i^{-1}(l + 1/2).$$

Clearly,  $A_i$  and  $B_i$  are  $n$ -faces of  $Q$ , and there are no other  $n$ -faces of  $Q$ . The usual boundary  $\text{bd}Q$  of  $Q$  is equal to the union of all faces  $A_i, B_i$ . The *big sphere* is

$$S^n = \bigcup_{i=1}^{n+1} A_i \cup B_i.$$

Obviously, the big sphere  $S^n$  is homeomorphic to the standard  $n$ -dimensional sphere and is invariant under the antipodal involution  $\iota$ .

**Cubes.** A *n-cube* of  $S^n$  is a product of the form

$$\prod_{i=1}^{n+1} J_i,$$

where each  $J_i$  is either an interval of the form  $[a - 1/2, a + 1/2]$ , where  $a$  is an integer between  $-l$  and  $l$ , or a boundary point of the interval  $[-l - 1/2, l + 1/2]$ , and exactly

one of the sets  $J_i$  is such a boundary point. Clearly, every  $n$ -cube is contained in  $S^n$  and  $n$ -cubes form a partition of  $S^n$  in an obvious sense. The *cubical sets* are defined as the unions of  $n$ -cubes. A *cube* is a product of the same form in which some of the sets  $J_i$  are allowed to be one-point sets  $\{b - 1/2\}$ , where  $b$  is an integer between  $-l$  and  $l + 1$ , and at least one of these numbers  $b$  is equal to  $-l$  or  $l + 1$ . The *dimension* of a cube and *m-cubes* are defined as before. The *m-chains* are formal sums of  $m$ -cubes with coefficients in  $\mathbb{F}_2$ . The *supports* and the *boundary operator*  $\partial$  are defined as before and  $\partial \circ \partial = 0$  as before.

Clearly, the antipodal involution  $\iota$  maps  $m$ -cubes to  $m$ -cubes,  $m$ -chains to  $m$ -chains etc. Two objects related by  $\iota$  are said to be *antipodal* to each other, and an object is called *symmetric* if  $\iota$  leaves it invariant. The image of a chain  $c$  under  $\iota$  is denoted by  $\iota_*(c)$ .

**Intersections with spheres of lower dimensions.** For  $k < n$  let  $S^k$  be the intersection of  $S^n$  with the subspace  $\mathbb{R}^{k+1} \times 0$  of  $\mathbb{R}^{n+1}$  defined by the equations  $x_i = 0$  with  $i \geq k + 2$ . Clearly, the intersection of an  $n$ -cube of  $S^n$  with  $S^k$  is either empty or a  $k$ -cube of  $S^k$ . More generally, the intersection of an  $(n - m)$ -cube in  $S^n$  with the sphere  $S^k$  is either empty or is a  $(k - m)$ -cube in  $S^k$  (for  $k < m$  it is always empty). Using the modern language, one can say that all cubes in  $S^n$  are in *general position* with respect to the sphere  $S^k$ . This wouldn't be the case if we would use an integer instead of  $l + 1/2$  in the definition of  $Q$ .

Let  $\gamma$  be an  $(n - m)$ -chain in  $S^n$ , i.e. let

$$\gamma = \sum_i c_i,$$

where  $c_i$  are  $(n - m)$ -cubes in  $S^n$ . The fact that all cubes in  $S^n$  are in general position with respect to  $S^k$  allows to define the *intersection*  $\gamma \cap S^k$  of  $\gamma$  with  $S^k$ . Namely,

$$\gamma \cap S^k = \sum_i c_i \cap S^k,$$

where the empty intersections are interpreted as 0's, is an  $(k - m)$ -chain in  $S^k$ . The operation of taking the intersection of chains with  $S^k$  commutes with  $\partial$ , i.e.

$$(\partial \gamma) \cap S^k = \partial (\gamma \cap S^k)$$

for any chain  $\gamma$  in  $S^n$ . This is trivial for cubes, and extends to the general case by linearity.

**5.1. Lemma.** *Every symmetric  $m$ -chain  $\gamma$  in  $S^n$  can be represented in the form*

$$(11) \quad \gamma = \delta + \iota_*(\delta),$$

where the  $m$ -chain  $\delta$  is such that  $\delta$  and  $\iota_*(\delta)$  have no common  $m$ -cubes. If  $\partial \gamma = 0$ , then  $\partial \delta$  is a symmetric  $(m - 1)$ -chain (which may be equal to zero).

**Proof.** Since  $\gamma$  is symmetric, the set of  $m$ -cubes of  $\gamma$  is invariant under the antipodal involution  $\iota$ . Since  $\iota$  leaves no  $m$ -cube invariant, this set is equal to the union of a collection of disjoint pairs of the form  $\{c, \iota(c)\}$ . If we select one  $m$ -cube from each such pair and define  $\delta$  as the sum of these selected cubes, then (11) obviously holds. If (11) holds, then

$$\partial\gamma = \partial\delta + \partial\iota_*(\delta) = \partial\delta + \iota_*(\partial\delta).$$

If, moreover,  $\partial\gamma = 0$ , then  $\partial\delta + \iota_*(\partial\delta) = 0$ . Since we are working over  $\mathbb{F}_2$ , this implies that  $\partial\delta = \iota_*(\partial\delta)$ , i.e.  $\partial\delta$  is symmetric. ■

**The odd 0-chains.** In what follows 0-chains will appear as the intersections  $\omega = \gamma \cap S^m$  of  $(n-m)$ -chains  $\gamma$  in  $S^n$  with  $S^m$ . Clearly, such a chain  $\omega$  is symmetric if  $\gamma$  is. If  $\omega$  is a symmetric 0-chain, then  $\omega$  is the sum of 0-cubes in a set of 0-cubes invariant under the action of  $\iota$ . Clearly, such a set is equal to the union of a collection of pairs of antipodal points (0-cubes). If the number of these pairs is odd, then  $\omega$  is said to be an *odd* 0-chain.

**5.2. Lemma.** *Let  $m \leq n$  and let  $\gamma$  be a symmetric  $(n-m)$ -chain in  $S^n$  such that  $\partial\gamma = 0$  and  $\gamma \cap S^m$  is an odd 0-chain. Suppose that  $\gamma$  is represented in the form*

$$\gamma = \delta + \iota_*(\delta)$$

*and the  $(n-m)$ -chains  $\delta$  and  $\iota_*(\delta)$  have no common  $(n-m)$ -cubes. Then*

$$\gamma' = \partial\delta$$

*is a symmetric  $(n-m-1)$ -chain and  $\gamma' \cap S^{m+1}$  is an odd 0-chain. In particular,  $\gamma' \neq 0$ .*

**Proof.** Lemma 5.1 implies that  $\gamma'$  is a symmetric  $(n-m-1)$ -chain. Since an odd 0-chain is obviously non-zero, it remains to prove that  $\gamma' \cap S^{m+1}$  is an odd 0-chain.

Let us think about the sphere  $S^m$  as the *equator* dividing  $S^{m+1}$  into two hemispheres, the *northern* one, defined by the inequality  $x_{m+2} \geq 0$  and denoted by  $N^{m+1}$ , and the *southern* one. The chain  $\gamma'$  is an  $(n-m-1)$ -chain and hence  $|\gamma'|$  is disjoint from  $S^m$ . Therefore it is sufficient to prove that  $|\gamma'| \cap N^{m+1}$  consists of an odd number of points.

The main idea is to consider the intersection of everything in sight with  $S^{m+1}$ . Let

$$\gamma_1 = \gamma \cap S^{m+1} \quad \text{and} \quad \delta_1 = \delta \cap S^{m+1}.$$

Then  $\gamma_1$  and  $\delta_1$  are 1-chains such that

$$\gamma_1 = \delta_1 + \iota_*(\delta_1),$$

the chains  $\delta_1$  and  $\iota_*(\delta_1)$  have no common 1-cubes,  $\gamma \cap S^m = \gamma_1 \cap S^m$ , and

$$\gamma' \cap S^{m+1} = \partial\delta \cap S^{m+1} = \partial\delta_1 \cap S^{m+1}.$$

In particular, it is sufficient to prove that the number of points in  $|\partial\delta_1| \cap N^{m+1}$  is odd.

Let  $\gamma_E$  be the sums of all 1-cubes of  $\gamma_1$  intersecting  $S^m$ . Then  $\gamma_E$  is a symmetric 1-chain and hence is the sum of 1-cubes in an  $\iota$ -invariant set. Clearly, such a set is equal to the union of a set of pairs of antipodal 1-cubes. Let  $p$  be the number of these pairs. Then the number of 1-cubes of  $\gamma_E$  is  $2p$  and  $p$  is equal to the number of pairs of antipodal points in

$$\gamma_E \cap S^m = \gamma_1 \cap S^m = \gamma \cap S^m.$$

Since  $\gamma_1 \cap S^m$  is an odd 0-chain,  $p$  is odd.

Let  $\delta_E$  be the sums of all 1-cubes of  $\delta_1$  intersecting  $S^m$ . Then

$$\gamma_E = \delta_E + \iota_*(\delta_E),$$

and the chains  $\delta_E$  and  $\iota_*(\delta_E)$  have no common 1-cubes. It follows that the number of 1-cubes of  $\delta_E$  is equal to the half of the number of 1-cubes of  $\gamma_E$ , i.e. to  $p$ .

Let  $\delta_N$  be the sums of all 1-cubes of  $\delta_1$  contained in the northern hemisphere  $N^{m+1}$ . Then  $\delta_E + \delta_N$  is the sum of all 1-cubes of  $\delta_1$  intersecting  $N^{m+1}$ , and hence

$$(12) \quad |\partial\delta_1| \cap N^{m+1} = |\partial(\delta_E + \delta_N)| \cap N^{m+1}$$

The 1-chain  $\partial(\delta_E + \delta_N)$  is a boundary and hence is a sum of an even number of points. Some of these points cancel under the summation, but this does not affect the parity of the number of points. Since  $|\delta_N|$  is contained inside of the northern hemisphere, the endpoints of 1-cubes of  $\delta_E$  contained in the southern hemisphere do not cancel. There are  $p$  such endpoints. Since  $p$  is odd, it follows that the number of points of  $|\partial(\delta_E + \delta_N)|$  contained in the northern hemisphere is odd. In view of (12) this implies that the number of points of  $|\partial\delta_1| \cap N^{m+1}$  is odd. As was noted above, this is sufficient to complete the proof. ■

**Distances.** Let us use as the distance in  $\mathbb{R}^{n+1}$  the  $l_\infty$ -distance defined as

$$\text{dist}(x, y) = \max_i |x_i - y_i|,$$

where  $x = (x_1, x_2, \dots, x_{n+1})$  and  $y = (y_1, y_2, \dots, y_{n+1})$ .

Obviously, if the points  $x, y$  belong to two disjoint cubes of  $S^n$ , then  $\text{dist}(x, y) \geq 1$ . It follows that the distance between two disjoint cubical subsets of  $S^n$  is  $\geq 1$ .

**5.3. Theorem.** Let  $e_1, e_2, \dots, e_n$  be cubical subsets of the sphere  $S^n$ . Suppose that none of them contains a pair of antipodal points. Then the sets  $e_i \cup \bar{e}_i$  do not cover  $S^n$ .

**Proof.** The set  $e_i$  is disjoint from  $\bar{e}_i$  for every  $i$ , and hence the distance between  $e_i$  and  $\bar{e}_i$  is  $\geq 1$  for every  $i$ . Suppose that the sets  $e_i \cup \bar{e}_i$  cover  $S^n$ .

Let us consider the images of  $S^n$  and the sets  $e_i$  under the map  $\mu: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  multiplying every coordinate by 3, i.e. under the map

$$\mu: (x_1, x_2, \dots, x_{n+1}) \mapsto (3x_1, 3x_2, \dots, 3x_{n+1}).$$

Since  $3(l + 1/2) = (3l + 1) + 1/2$ , the image  $\mu(S^n)$  is the sphere constructed with the number  $3l + 1$  in the role of  $l$ . Obviously, the images  $\mu(e_i)$  of the sets  $e_i$  are disjoint from their antipodal sets, and it is sufficient to prove that the sets  $\mu(e_i) \cup \mu(\bar{e}_i)$  do not cover  $\mu(S^n)$ . Hence we can replace  $l$  by  $3l + 1$  in the definition of  $S^n$  and the sets  $e_i$  by their images  $\mu(e_i)$ . We will use for the new sets  $\mu(e_i)$  the old notation  $e_i$ .

Now the distance between  $e_i$  and  $\bar{e}_i$  is  $\geq 3$  for every  $i$ , while the cubes have the same size as before. For every  $i$  let  $E_i$  be the union of  $e_i$  and all  $n$ -cubes intersecting  $e_i$ . Then  $e_i$  is contained in the interior of  $E_i$ . Clearly, the distance of every point of  $E_i$  from  $e_i$  is  $\leq 1$ , and the distance of every point of  $\bar{E}_i$  from  $\bar{e}_i$  is also  $\leq 1$ . Since the distance between  $e_i$  and  $\bar{e}_i$  is  $\geq 3$ , it follows that  $E_i$  is disjoint from  $\bar{E}_i$  for every  $i$ .

Let us recursively construct for each  $m = 0, 1, \dots, n$  a symmetric  $(n - m)$ -chain  $\gamma_m$  in  $S^n$  such that  $\partial\gamma_m = 0$ , the 0-chain  $\gamma_m \cap S^m$  is odd, and the support  $|\gamma_m|$  is disjoint from  $e_i \cup \bar{e}_i$  for  $1 \leq i \leq m$  after a possible renumbering of the sets  $e_i$ .

Let  $\gamma_0 = \llbracket S^n \rrbracket$ , where  $\llbracket S^n \rrbracket$  is the sum of all  $n$ -cubes in  $S^n$ . Clearly,  $\partial\gamma_0 = 0$ , the 0-chain  $\gamma_0 \cap S^0$  is equal to  $\llbracket S^0 \rrbracket$  and hence is odd, and the condition concerned with  $|\gamma_0|$  is vacuous. Therefore  $\gamma_0$  has the required properties.

Suppose that  $\gamma_m$  is already constructed. Since the sets  $e_i \cup \bar{e}_i$  are covering  $S^n$  and are disjoint from  $|\gamma_m|$  for  $i \leq m$ , there exists  $j \geq m + 1$  such that  $e_j \cup \bar{e}_j$  has non-empty intersection with  $|\gamma_m|$ . After renumbering the sets  $e_j$  with  $j \geq m + 1$  we may assume that  $e_{m+1} \cup \bar{e}_{m+1}$  has non-empty intersection with  $|\gamma_m|$ . Since  $\gamma_m$  is symmetric, this implies that  $e_{m+1} \cap |\gamma_m| \neq \emptyset$  and hence  $E_{m+1} \cap |\gamma_m| \neq \emptyset$ . Let  $D_m$  be the closure of

$$|\gamma_m| \setminus (E_{m+1} \cup \bar{E}_{m+1}).$$

Clearly,  $D_m$  is the union of a symmetric set of  $(n - m)$ -cubes. Let us select one  $(n - m)$ -cube from each pair of the form  $c, \bar{c}$  contained in this set of cubes and consider the union  $F_{m+1}$  of  $E_{m+1} \cap |\gamma_m|$  with these selected  $(n - m)$ -cubes. Then  $F_{m+1} \cup \bar{F}_{m+1} = |\gamma_m|$ , and since  $E_{m+1}$  is disjoint from  $\bar{E}_{m+1}$ , the set  $F_{m+1}$  has no common  $(n - m)$ -cubes with

$\bar{F}_{m+1}$ . Let  $\delta_m$  be the sum of all  $(n-m)$ -cubes contained in  $F_{m+1}$ . Then

$$\gamma_m = \delta_m + \iota_*(\delta_m)$$

and the  $(n-m)$ -chains  $\delta_m$  and  $\iota_*(\delta_m)$  have no common  $(n-m)$ -cubes. Let

$$\gamma_{m+1} = \partial\delta_m.$$

By Lemma 5.2 the  $(n-m-1)$ -chain  $\gamma_{m+1}$  is symmetric and  $\gamma_{m+1} \cap S^{m+1}$  is an odd 0-chain. The identity  $\partial \circ \partial = 0$  implies that  $\partial\gamma_{m+1} = 0$ . Clearly,  $|\gamma_{m+1}| \subset |\gamma_m|$  and hence  $|\gamma_{m+1}|$  is disjoint from  $e_i \cup \bar{e}_i$  for  $i \leq m$ . By the construction,  $|\gamma_{m+1}|$  may intersect the set  $E_{m+1} \cup \bar{E}_{m+1}$  only along its boundary. Since  $e_{m+1} \cup \bar{e}_{m+1}$  is contained in its interior,  $|\gamma_{m+1}|$  is disjoint from  $e_{m+1} \cup \bar{e}_{m+1}$  also. Therefore  $\gamma_{m+1}$  has the required properties.

This completes the construction of chains  $\gamma_m$ . The last of them is the 0-chain  $\gamma_n$ . Its support  $|\gamma_n|$  is disjoint from all sets  $e_i \cup \bar{e}_i$ . At the same time  $|\gamma_n|$  is non-empty because  $\gamma_n = \gamma_n \cap S^n$  is an odd 0-chain. It follows that the sets  $e_i \cup \bar{e}_i$  do not cover  $S^n$ . ■

**The first Lusternik–Schnirelmann theorem.** *Let  $S^n$  be the standard unit sphere in  $\mathbb{R}^{n+1}$ . Let  $F_1, F_2, \dots, F_n$  be closed subsets of  $S^n$ . Suppose that none of them contains a pair of antipodal points. Then the sets  $F_i \cup \bar{F}_i$  do not cover  $S^n$ .*

**Proof.** For every  $i$  the sets  $F_i$  and  $\bar{F}_i$  are disjoint. Since they are closed subsets  $S^n$  and  $S^n$  is compact, the distance between them is  $> 0$ . Let  $\varepsilon > 0$  be a real number smaller than the distance between  $F_i$  and  $\bar{F}_i$  for every  $i$ .

Let  $r: S^n \rightarrow S^n$  be the radial projection. The images  $r(c)$  of  $n$ -cubes of  $S^n$  form a partition of the sphere  $S^n$ . Let  $E_i$  be the union of all such images intersecting  $F_i$ . Clearly,  $F_i \subset E_i$ . If  $l$  is sufficiently big, then the diameter of  $r(c)$  is  $< \varepsilon/3$  for every  $c$ . In this case the distance of every point of  $E_i$  from  $F_i$  is  $< \varepsilon/3$ , and hence  $E_i$  and  $\bar{E}_i$  are disjoint.

For every  $i$  let  $e_i = r^{-1}(E_i)$ . Then the sets  $e_i$  are cubical sets, and for every  $i$  the sets  $e_i$  and  $\bar{e}_i$  are disjoint. By Theorem 5.3 the sets  $e_i \cup \bar{e}_i$  do not cover  $S^n$ . It follows that the sets  $E_i \cup \bar{E}_i$  do not cover  $S^n$ , and hence their subsets  $F_i \cup \bar{F}_i$  also do not cover  $S^n$ . ■

**The second Lusternik–Schnirelmann theorem.** *Let  $F_1, F_2, \dots, F_{n+1}$  be closed subsets of  $S^n$ . If none of them contains a pair of antipodal points, then these sets do not cover  $S^n$ .*

**Proof.** By the previous theorem the sets  $F_i \cup \bar{F}_i$  with  $i \leq n$  do not cover  $S^n$ , i.e. the complement  $C$  of their union is non-empty. If the sets  $F_1, F_2, \dots, F_{n+1}$  cover  $S^n$ , then  $C$  is contained in  $F_{n+1}$ . But  $C$  is obviously invariant under the antipodal involution, and hence in this case  $F_{n+1}$  contains a pair of antipodal points, contrary to the assumption. ■



## 6. First applications of Lusternik–Schirelmann theorems

**6.1. Theorem.** *If  $m < n$ , then there exists no continuous map  $f: \mathbb{S}^n \rightarrow \mathbb{S}^m$  taking pairs of antipodal points to pairs of antipodal points, i.e. such that  $f(-x) = -f(x)$  for all  $x$ .*

**Proof.** Let  $f: \mathbb{S}^n \rightarrow \mathbb{S}^m$  be a continuous map. Let  $\varepsilon > 0$  be a small number. For every  $i = 1, 2, \dots, m+1$  let  $H_i$  be the subset of  $\mathbb{S}^m$  defined by the inequality  $x_i \geq \varepsilon$ , where  $x_1, x_2, \dots, x_{m+1}$  are the standard coordinates in  $\mathbb{R}^{m+1}$ . Clearly,  $H_i$  is disjoint from  $\bar{H}_i$ . If  $\varepsilon$  is small enough, then the sets  $H_i \cup \bar{H}_i$  cover  $\mathbb{S}^m$  and hence their preimages

$$f^{-1}(H_i \cup \bar{H}_i) = f^{-1}(H_i) \cup f^{-1}(\bar{H}_i)$$

cover  $\mathbb{S}^n$ . Also, the preimages of  $H_i$  and  $\bar{H}_i$  are disjoint because  $H_i$  and  $\bar{H}_i$  are.

Let  $F_i = f^{-1}(H_i)$ . If  $f$  takes pairs of antipodal points to pairs of antipodal points, then

$$f^{-1}(\bar{H}_i) = \bar{F}_i$$

for every  $i$ . It follows that  $F_i$  is a closed set disjoint from  $\bar{F}_i$  for every  $i$  and the sets  $F_i \cup \bar{F}_i$  cover  $\mathbb{S}^n$ . But there are  $m+1 \leq n$  sets  $F_i$  and hence we reached a contradiction with the first Lusternik–Schnirelmann theorem (see Section 5). Therefore no such map  $f$  exists. ■

**Borsuk–Ulam theorem.** *Let  $f: \mathbb{S}^n \rightarrow \mathbb{R}^n$  be a continuous map. Then there is a point  $x \in \mathbb{S}^n$  such that  $f(-x) = f(x)$  for some  $x$ .*

**Proof.** Suppose that  $f(-x) \neq f(x)$  for all  $x \in \mathbb{S}^n$ . Then

$$g: x \mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|},$$

where  $\|\bullet\|$  is the usual euclidean norm, is a well defined continuous map  $\mathbb{S}^n \rightarrow \mathbb{S}^{n-1}$ . Obviously,  $g(-x) = -g(x)$  for all  $x \in \mathbb{S}^n$ , in contradiction with Theorem 6.1. ■

**Borsuk Theorem.** *If a continuous map  $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$  takes pairs of antipodal points to pairs of antipodal points, then  $f$  cannot be extended to a map  $\mathbb{B}^{n+1} \rightarrow \mathbb{S}^n$ .*

**Proof.** Suppose that  $g: \mathbb{B}^{n+1} \rightarrow \mathbb{S}^n$  is such an extension. Let  $p: \mathbb{S}^{n+1} \rightarrow \mathbb{B}^{n+1}$  be the projection forgetting the last coordinate. Let us define a map  $h: \mathbb{S}^{n+1} \rightarrow \mathbb{S}^n$  as follows:  $h(x) = g \circ p(x)$  if  $x$  is in the northern hemisphere of  $\mathbb{S}^{n+1}$  and  $h(x) = -g \circ p(-x)$  if in the southern. Both formulas agree on  $\mathbb{S}^n$  and hence  $h$  is well defined. Clearly,  $h$  takes antipodal points to antipodal points, in contradiction with Theorem 6.1. ■

**6.2. Theorem.** *Let  $F_1, F_2, \dots, F_{n+1}$  be closed subsets of  $\mathbb{S}^n$ . Suppose that none of them contains a pair of antipodal points and the sets  $F_i \cup \bar{F}_i$  cover  $\mathbb{S}^n$ . Then the intersection of the sets  $F_1, F_2, \dots, F_{n+1}$  is non-empty.*

**Proof.** Since the sets  $F_i$  and  $\bar{F}_i$  are closed and disjoint, there are continuous functions

$$f_i : \mathbb{S}^n \longrightarrow [-1/2, 1/2]$$

such that  $f_i^{-1}(1/2) = F_i$  and  $f_i^{-1}(-1/2) = \bar{F}_i$ . Let  $g_i(x) = f_i(x) - f_i(-x)$ . Then  $g_i$  is a continuous function  $\mathbb{S}^n \longrightarrow [-1, 1]$  such that  $g_i^{-1}(1) = F_i$  and  $g_i^{-1}(-1) = \bar{F}_i$ . In addition,  $g_i(-x) = -g_i(x)$  for all  $x \in \mathbb{S}^n$ . Together the functions  $g_i$  define a map

$$g : \mathbb{S}^n \longrightarrow [-1, 1]^{n+1}$$

such that  $g(-x) = -g(x)$  for all  $x \in \mathbb{S}^n$ . Since the sets  $F_i \cup \bar{F}_i$  cover  $\mathbb{S}^n$ , the image of  $g$  is contained in the boundary  $\mathbf{S}^n$  of the cube  $[-1, 1]^{n+1}$ .

Suppose that  $F_1 \cap F_2 \cap \dots \cap F_{n+1} = \emptyset$ . Then also  $\bar{F}_1 \cap \bar{F}_2 \cap \dots \cap \bar{F}_{n+1} = \emptyset$ . It follows that the points  $\mathbb{1} = (1, 1, \dots, 1)$  and  $-\mathbb{1} = (-1, -1, \dots, -1)$  are not in the image of  $g$ .

As we will see in a moment, there is a continuous map

$$q : \mathbf{S}^n \setminus \{\mathbb{1}, -\mathbb{1}\} \longrightarrow \mathbb{S}^{n-1}$$

such that  $q(-x) = -q(x)$  for all  $x \in \mathbf{S}^n \setminus \{\mathbb{1}, -\mathbb{1}\}$ . If  $q$  is such a map, then

$$h : x \longmapsto q(g(x))$$

is a continuous map  $\mathbb{S}^n \longrightarrow \mathbb{S}^{n-1}$  such that  $h(-x) = -h(x)$  for all  $x \in \mathbb{S}^n$ . The contradiction with Corollary 6.1 completes the proof modulo the existence of the map  $q$ .

Let  $\mathbb{1}^\perp$  be the  $n$ -dimensional vector subspace of  $\mathbb{R}^{n+1}$  consisting of vectors orthogonal to  $\mathbb{1}$ , and let  $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{1}^\perp$  be the orthogonal projection to  $\mathbb{1}^\perp$ . Then  $p(-x) = -p(x)$  for all  $x$  and  $p(x) = 0$  if and only if  $x$  is proportional to  $\mathbb{1}$ . In particular,  $p(x) \neq 0$  if  $x \in \mathbf{S}^n \setminus \{\mathbb{1}, -\mathbb{1}\}$ . It follows that the formula

$$q_1(x) = \frac{p(x)}{\|p(x)\|},$$

where  $\|\bullet\|$  is the usual euclidean norm, defines a map from  $\mathbf{S}^n \setminus \{\mathbb{1}, -\mathbb{1}\}$  to the unit sphere of  $\mathbb{1}^\perp$  such that  $q_1(-x) = -q_1(x)$  for all  $x$ . In order to get the map  $q$ , it remains to identify this unit sphere with  $\mathbb{S}^{n-1}$  by a rotation. ■

**6.3. Theorem.** Let  $f: \mathbb{S}^n \rightarrow \mathbb{R}^n$  be a continuous map. If  $f(-x) = -f(x)$  for every  $x \in \mathbb{S}^n$ , then  $f(x) = 0$  for some  $x \in \mathbb{S}^n$ .

**Proof.** If  $f(x) \neq 0$  for every  $x \in \mathbb{S}^n$ , then  $x \mapsto f(x)/\|f(x)\|$  is a continuous map  $\mathbb{S}^n \rightarrow \mathbb{S}^{n-1}$  contradicting Theorem 6.1. ■

**Historical remarks.** The applications of Lusternik-Schnirelmann theorems discussed in this section are the simplest ones and are the first only in a logical (or an expository) sense. Lusternik and Schnirelmann proved their theorems [LS] with applications to variational problems in mind. Their main result is a solution of a problem of Poincaré about geodesics. Namely, Lusternik and Schnirelmann proved [LS] that on every closed convex surface in  $\mathbb{R}^3$  there are at least three closed geodesics without self-intersections. They did not state the theorem which we called the first Lusternik–Schnirelmann theorem, but its proof is the main part of their proof of the theorem which we called the second Lusternik–Schnirelmann theorem. The latter is Lemma 1 in Section II.5 of [LS].

The paper [Bo] of K. Borsuk, published in 1933, three years after Lusternik–Schnirelmann book [LS], is devoted to three theorems about the topology of spheres. His main theorem, from which he deduced the two other, is the theorem called *Borsuk theorem* above. Strangely enough, his second theorem became known as *Borsuk–Ulam theorem*. Borsuk mentioned in a footnote in [Bo] that this result was suggested by S. Ulam as a conjecture. Ulam wasn't involved in the proof. G.-C. Rota [R] tells this story in his unique poetic style mixed with bitterness. The third Borsuk's theorem is the second Lusternik–Schnirelmann theorem from Section 5. Apparently, Borsuk was unaware of the work of Lusternik and Schnirelmann, and his methods are completely different. In a footnote Borsuk wrote

Herr H. Hopf, to whom I communicated the first theorem, briefly indicated to me three other short proofs of these theorems. But these proofs are based on deep results of the theory of degrees of maps and my proof basically is quite elementary, and hence I believe that it is not superfluous.

Borsuk's main tools are homotopies and, this footnote notwithstanding, the degrees of maps. In [Bo] he build by bare hands a fragment of the homotopy theory of maps to spheres. One may speculate that at least one of Hopf's proofs was a high-brow version of Borsuk's proof. For a modern elementary proof in this spirit see [Ma], Section 2.2.

Strangely enough, Theorem 6.2 is not so well known as the other versions of Borsuk–Ulam theorem. J. Matoušek included in [Ma], Section 2.1 about a dozen versions, but not Theorem 6.2. The author learned Theorem 6.2 from Lefschetz's book [Le]. See [Le], Section IV.7, Theorem (21.1). Apparently, this theorem is due to A.W. Tucker. He proved the case  $n = 2$  in [T1]. In full generality his results were published only in [Le]. A similar result was proved by Alexandroff and Hopf [AH]. See [AH], Section XII.3, Theorem X. Alexandroff and Hopf (ibid., Theorem VIII) write that Theorem 6.3 was suggested to them by G. Pólya.

## 7. The discrete cube and products of cubical cochains

**The discrete cube.** As usual, let  $n$  be a natural number and  $I = \{1, 2, \dots, n\}$ . Let  $k$  be another natural number. The *discrete cube* of size  $k$  is

$$K = \{0, 1, \dots, k\}^n.$$

For  $i = 1, 2, \dots, n$  let  $\mathcal{A}_i$  and  $\mathcal{B}_i$  be the sets of all points

$$(a_1, a_2, \dots, a_n) \in K$$

such that  $a_i = 0$  and  $a_i = k$  respectively. An  $(n-1)$ -*face* of  $K$  is a subset of  $K$  equal to  $\mathcal{A}_i$  or  $\mathcal{B}_i$  for some  $i$ . An interested reader may define  $m$ -faces of  $K$  with  $m \neq n-1$ . The faces  $\mathcal{A}_i, \mathcal{B}_i$ , where  $i \in I$ , are said to be *opposite*. The *boundary*  $\text{bd} K$  is the union of the  $(n-1)$ -faces.

**The cubes of  $K$ .** An  $n$ -*cube* of  $K$  is a product of the form

$$\prod_{i=1}^n \{a_i, a_i + 1\},$$

where  $a_1, \dots, a_n$  are integers between 0 and  $k-1$ . A *cube* of  $K$  is a product of the form

$$(13) \quad \sigma = \prod_{i=1}^n \rho_i,$$

where for every  $i \in I$  either  $\rho_i = \{a_i, a_i + 1\}$  for a non-negative integer  $a_i \leq k-1$ , or  $\rho_i = \{a_i\}$  for a non-negative integer  $a_i \leq k$ . The *dimension* of  $\sigma$  is the number of pairs  $\{a_i, a_i + 1\}$  in the product (13).

An  $m$ -*cube* is a cube of dimension  $m$ . A  $m$ -*face* of a cube  $\sigma$  is an  $m$ -cube contained in  $\sigma$ . Every  $m$ -face of the cube (13) is obtained by keeping  $m$  pairs  $\{a_i, a_i + 1\}$  in the product (13) intact and replacing other such pairs by one of their elements.

**Chains and cochains of  $K$ .** Now one can define  $m$ -chains and  $m$ -cochains of  $K$  in the same manner as before. The  $m$ -*chains* of  $K$  are defined as formal sums of  $m$ -cubes of  $K$  with coefficients in  $\mathbb{F}_2$ . Clearly, they form a vector space over  $\mathbb{F}_2$  (of course, the  $m$ -chains of  $Q$  also form such a vector space). Let us denote this vector space by  $C_m(K)$ . The *cubical  $m$ -cochains* of  $K$ , or simply the  $m$ -*cochains*, are defined as the elements of the dual vector space  $C^m(K) = C_m(K)^*$ . Since the vector space  $C_m(K)$  has a canonical basis, namely, the basis consisting of  $m$ -cubes of  $K$ , one can identify cochains with the formal sums of  $m$ -

cubes, similarly to the usual cochains. But viewing  $m$ -cochains as  $\mathbb{F}_2$ -valued functions on the set of  $m$ -cubes of  $K$  turns out to be more convenient.

The boundary  $\partial\sigma$  of an  $m$ -cube  $\sigma$  is defined as the formal sum of  $(m-1)$ -faces of  $\sigma$ . The map  $\sigma \longrightarrow \partial\sigma$  extends by linearity to the *boundary operator*

$$\partial: C_m(K) \longrightarrow C_{m-1}(K).$$

By the same reasons as before,  $\partial \circ \partial = 0$ . The *coboundary operator*

$$\partial^*: C^{m-1}(K) \longrightarrow C^m(K)$$

is the linear operator dual to  $\partial$ . Clearly,  $\partial^* \circ \partial^* = 0$ . If  $\tau$  is an  $(m-1)$ -cube, then

$$\partial^*(\tau) = \sum \sigma,$$

where the sum is taken over all  $m$ -cubes  $\sigma$  having  $\tau$  as a face.

**Products of cochains.** Let  $\sigma$  be the cube given by the product (13) with the sets  $\rho_i$  as above. The *direction* of  $\sigma$  is defined as the set  $A(\sigma)$  of subscripts  $i \in I$  such that  $\rho_i$  has the form  $\{a_i, a_i + 1\}$ . If  $\sigma$  is an  $m$ -cube, then  $A(\sigma)$  is an  $m$ -element set.

Suppose that  $H \subset A(\sigma)$  and  $\varepsilon = 0$  or  $1$ . Let us replace in the product (13) every factor  $\rho_i = \{a_i, a_i + 1\}$  with  $i \in H$  by  $\{a_i + \varepsilon\}$ . The resulting product is denoted by

$$\lambda_H^\varepsilon \sigma.$$

If  $\sigma$  is an  $m$ -cube and  $H$  is an  $h$ -element set, then  $\lambda_H^\varepsilon \sigma$  is an  $(m-h)$ -cube. In these terms

$$\partial\sigma = \sum_{P, \varepsilon} \lambda_P^\varepsilon \sigma,$$

where  $P$  runs over 1-element subsets of  $A(\tau)$  and  $\varepsilon$  runs over  $\{0, 1\}$ .

The following definition is an adaptation of the definition of products of singular cubical cochains due to J.-P. Serre [Se]. Let us consider  $m$ -cochains as  $\mathbb{F}_2$ -valued functions on the set of  $m$ -cubes. Suppose that  $f$  and  $g$  are  $p$ -cochain and  $q$ -cochain respectively. The *product*  $f \cdot g$  is the  $(p+q)$ -cochain defined by the formula

$$(14) \quad f \cdot g(\sigma) = \sum_{F, G} f(\lambda_G^0 \sigma) \cdot g(\lambda_F^1 \sigma),$$

where  $\sigma$  is an  $(p+q)$ -cube, and  $F, G$  runs over all pairs of disjoint subsets of  $A(\sigma)$  consisting of  $p$  and  $q$  elements respectively. Equivalently,  $F$  runs over all  $p$ -element subsets of  $A(\sigma)$ , and  $G$  is the complement of  $F$  in  $A(\sigma)$ , i.e.  $G = A(\sigma) \setminus F$ .

**7.1. Lemma.** *The product of cochains is associative.*

**Proof.** Suppose that  $f$ ,  $g$  and  $h$  are  $p$ -cochain,  $q$ -cochain, and  $r$ -cochain respectively, and that  $\sigma$  is an  $(p + q + r)$ -cube. Then

$$(f \cdot g) \cdot h(\sigma) = \sum_{F, G, H} f(\lambda_G^0 \lambda_H^0 \sigma) \cdot g(\lambda_F^1 \lambda_H^0 \sigma) \cdot h(\lambda_{F \cup G}^1 \sigma),$$

where  $F, G, H$  runs over triples of disjoint subsets of  $A(\sigma)$  consisting of  $p$ ,  $q$  and  $r$  elements respectively. Similarly,

$$f \cdot (g \cdot h)(\sigma) = \sum_{F, G, H} f(\lambda_{G \cup H}^0 \sigma) \cdot g(\lambda_H^0 \lambda_F^1 \sigma) \cdot h(\lambda_G^1 \lambda_F^1 \sigma),$$

where  $F, G, H$  runs over the same triples. The two sums are equal because, obviously,

$$\lambda_G^0 \lambda_H^0 \sigma = \lambda_{G \cup H}^0 \sigma, \quad \lambda_{F \cup G}^1 \sigma = \lambda_G^1 \lambda_F^1 \sigma,$$

and

$$\lambda_F^1 \lambda_H^0 \sigma = \lambda_H^0 \lambda_F^1 \sigma.$$

The lemma follows. ■

**7.2. Lemma (Leibniz formula).**  $\partial^*(f \cdot g) = (\partial^* f) \cdot g + f \cdot (\partial^* g)$  for every cochains  $f, g$ .

**Proof.** Suppose that  $f$  and  $g$  are  $p$ -cochain and  $q$ -cochain respectively, and that  $\sigma$  is an  $(p + q + 1)$ -cube. Then

$$\partial^*(f \cdot g)(\sigma) = \sum_{F, G, P, \varepsilon} f(\lambda_G^0 \lambda_P^\varepsilon \sigma) \cdot g(\lambda_F^1 \lambda_P^\varepsilon \sigma),$$

where  $F, G, P$  runs over triples of disjoint subsets of  $A(\sigma)$  consisting of  $p$ ,  $q$  and 1 elements respectively, and  $\varepsilon$  runs over  $\{0, 1\}$ . Similarly,

$$(\partial^* f) \cdot g(\sigma) = \sum_{F, G, P, \varepsilon} f(\lambda_P^\varepsilon \lambda_G^0 \sigma) \cdot g(\lambda_{F \cup P}^1 \sigma),$$

and

$$f \cdot (\partial^* g)(\sigma) = \sum_{F, G, P, \varepsilon} f(\lambda_{G \cup P}^0 \sigma) \cdot g(\lambda_P^\varepsilon \lambda_F^1 \sigma),$$

where  $F, G, P$  and  $\varepsilon$  run over the same sets.

It follows that  $(\partial^* f) \cdot g(\sigma)$  is equal to

$$(15) \quad \sum_{F, G, P} f(\lambda_P^0 \lambda_G^0 \sigma) \cdot g(\lambda_{F \cup P}^1 \sigma) + \sum_{F, G, P} f(\lambda_P^1 \lambda_G^0 \sigma) \cdot g(\lambda_{F \cup P}^1 \sigma)$$

and  $f \cdot (\partial^* g)(\sigma)$  is equal to

$$(16) \quad \sum_{F, G, P} f(\lambda_{G \cup P}^0 \sigma) \cdot g(\lambda_P^1 \lambda_F^1 \sigma) + \sum_{F, G, P} f(\lambda_{G \cup P}^0 \sigma) \cdot g(\lambda_P^0 \lambda_F^1 \sigma)$$

Since the sets  $F$ ,  $G$  and  $P$  are disjoint,

$$\lambda_P^0 \lambda_G^0 \sigma = \lambda_{G \cup P}^0 \sigma$$

and

$$\lambda_{F \cup P}^1 \sigma = \lambda_P^1 \lambda_F^1 \sigma,$$

and hence the first sums in (15) and (16) are equal and cancel under addition. Therefore

$$\begin{aligned} & (\partial^* f) \cdot g(\sigma) + f \cdot (\partial^* g)(\sigma) \\ &= \sum_{F, G, P} f(\lambda_P^1 \lambda_G^0 \sigma) \cdot g(\lambda_{F \cup P}^1 \sigma) + \sum_{F, G, P} f(\lambda_{G \cup P}^0 \sigma) \cdot g(\lambda_P^0 \lambda_F^1 \sigma) \\ &= \sum_{F, G, P} f(\lambda_P^1 \lambda_G^0 \sigma) \cdot g(\lambda_P^1 \lambda_F^1 \sigma) + \sum_{F, G, P} f(\lambda_P^0 \lambda_G^0 \sigma) \cdot g(\lambda_P^0 \lambda_F^1 \sigma) \\ &= \sum_{F, G, P, \varepsilon} f(\lambda_G^0 \lambda_P^\varepsilon \sigma) \cdot g(\lambda_F^1 \lambda_P^\varepsilon \sigma). \end{aligned}$$

This is nothing else but the above expression for  $\partial^*(f \cdot g)(\sigma)$ . The lemma follows. ■

**Another definition of products.** The associativity implies that products  $f \cdot g \cdot \dots \cdot h$  of several cochains are well-defined. Now we will rephrase the definition (14) in a form more suitable for working with multiple products. Given a cube  $\sigma$ , let  $A = A(\sigma)$ , and let

$$\lambda^0 \sigma = \lambda_A^0 \sigma \quad \text{and} \quad \lambda^1 \sigma = \lambda_A^1 \sigma.$$

Both  $\lambda^0 \sigma$  and  $\lambda^1 \sigma$  are 0-cubes, which we will call respectively the *root* and the *peak* of  $\sigma$ .

In these terms the definition of the product takes the following form. Suppose that  $f$  and  $g$  are  $p$ -cochain and  $q$ -cochain respectively, and let  $\sigma$  be a  $(p + q)$ -cube. Then

$$(17) \quad f \cdot g(\sigma) = \sum_{\tau, \rho} f(\tau) \cdot g(\rho),$$

where the sum is taken over all pairs  $\tau, \rho$  such that  $\tau$  is a  $p$ -face of  $\sigma$  and  $\rho$  is a  $q$ -face of  $\sigma$ , the directions of  $\tau$  and  $\rho$  are disjoint, the root of  $\tau$  is equal to the root of  $\sigma$ , and the peak of  $\tau$  is equal to the root of  $\rho$ , i.e.

$$A(\tau) \cap A(\rho) = \emptyset,$$

$$\lambda^0 \tau = \lambda^0 \sigma, \quad \text{and} \quad \lambda^1 \tau = \lambda^0 \rho$$

Indeed, suppose that  $\sigma$  is given by (13). If  $F, G$  are disjoint subsets of  $A(\sigma)$  as in (14), then  $\tau = \lambda_G^0 \sigma$  and  $\rho = \lambda_F^1 \sigma$  are faces of  $\sigma$  with directions  $F$  and  $G$  respectively, and

$$\lambda^1 \tau = \lambda^0 \rho = (a_1 + \varepsilon_1, a_2 + \varepsilon_2, \dots, a_n + \varepsilon_n),$$

where  $\varepsilon_i = 0$  if  $i \notin A(\sigma)$  or  $i \in G$ , and  $\varepsilon_i = 1$  if  $i \in F$ . Conversely, if  $\tau, \rho$  have the above properties, then  $\tau = \lambda_G^0 \sigma$  and  $\rho = \lambda_F^1 \sigma$ , where  $F = A(\tau)$  and  $G = A(\rho)$ . It follows that the definitions (14) and (17) are equivalent.

**Products of several cochains.** The second formula for the product, i.e. the formula (17), admits a transparent extension to multiple products. Suppose that  $f_i$  is a  $p_i$ -cochain for  $i = 1, 2, \dots, r$ . If  $\sigma$  is a  $(p_1 + p_2 + \dots + p_r)$ -cube, then

$$(18) \quad f_1 \cdot f_2 \cdot \dots \cdot f_r(\sigma) = \sum f_1(\tau_1) \cdot f_2(\tau_2) \cdot \dots \cdot f_r(\tau_r),$$

where the sum is taken over all sequences  $\tau_1, \tau_2, \dots, \tau_r$  such that  $\tau_i$  is a  $p_i$ -face of  $\sigma$ , the directions of these faces are pair-wise disjoint, the root of  $\tau_1$  is equal to the root of  $\sigma$ , and the peak of  $\tau_i$  is equal to the root of  $\tau_{i+1}$  for all  $i \leq r - 1$ .

A sequence  $\tau_1, \tau_2, \dots, \tau_r$  of faces of  $\sigma$  subject to the above conditions is uniquely determined by the directions of the cubes  $\tau_i$ . Indeed, a cube is uniquely determined by its root and direction. Therefore, the above conditions allows to find consecutively the cubes  $\tau_1, \tau_2, \dots, \tau_r$  if their directions are known.

**Products of 1-cochains.** We are especially interested in the case when  $r = n$  and every  $f_i$  is a 1-cochain. Then every  $\tau_i$  in (18) is a 1-cube and its direction is a 1-element subset of  $I$ . In this case the sequences of directions of cubes  $\tau_i$ , and hence the sequences



$\tau_1, \tau_2, \dots, \tau_n$  themselves are in an obvious one-to-one correspondence with the permutations of the set  $I = \{1, 2, \dots, n\}$ . Also, every such sequence  $\tau_1, \tau_2, \dots, \tau_n$  has the form

$$\tau_i = \{v_{i-1}, v_i\}$$

for a sequence

$$v_0, v_1, \dots, v_n \in \sigma$$

such that the 0-cube  $\{v_0\}$  is the root of  $\sigma$ , the 0-cube  $\{v_n\}$  is the peak of  $\sigma$ , every pair  $\{v_{i-1}, v_i\}$  is a 1-cube, and the directions of these 1-cubes are all different. We already encountered such sequences  $v_0, v_1, \dots, v_n$ . As one can easily check, they are nothing else but the pivot sequences from Section 2. But the reader may ignore this fact and treat the above description as the definition of *pivot sequences*.

If  $v_0, v_1, \dots, v_n$  is a pivot sequence, passing from  $v_{i-1}$  to  $v_i$  increases one of the coordinates by 1 and leaves other coordinates unchanged. In particular, no coordinate decreases along a pivot sequence. Also, no pair  $\{v_i, v_j\}$  with  $|i - j| \geq 2$  is a 1-cube. It follows that a pivot sequence  $v_0, v_1, \dots, v_n$  is uniquely determined by the set  $\{v_0, v_1, \dots, v_n\}$ .

**7.3. Lemma.** *Let  $f$  be an  $(n - 1)$ -cochain. Then*

$$\sum_{\sigma} \partial^* f(\sigma) = \sum_{\tau} f(\tau),$$

where the first sum is taken over all  $n$ -cubes  $\sigma$  of  $K$  and the second sum is taken over all  $(n - 1)$ -cubes  $\tau$  contained in the boundary  $\text{bd} K$ .

**Proof.** Let us consider cochains as formal sums of cubes. Clearly, it is sufficient to prove the lemma in the case when  $f$  is equal to an  $(n - 1)$ -cube  $\tau$ . If  $\tau$  is contained in  $\text{bd} K$ , then  $\tau$  is a face of exactly one  $n$ -cube, and both sums are equal to 1. If  $\tau$  is not contained in  $\text{bd} K$ , then  $\tau$  is a face of exactly two  $n$ -cubes. In this case the left sum is equal to 2 and the right sum is empty and hence is equal to 0. Since  $2 = 0$  in  $\mathbb{F}_2$ , the lemma follows. ■

**7.4. Theorem.** *Let  $h_1, h_2, \dots, h_n$  be 0-cochains of  $K$  and let  $f_i = \partial^* h_i$  for every  $i \in I$ . Let  $s$  be the number of  $n$ -cubes  $\sigma$  of  $K$  such that*

$$f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1.$$

*Let  $t$  be the number of  $(n - 1)$ -cubes  $\tau \subset \text{bd} K$  such that  $h_1(\lambda^0 \tau) = 1$  and*

$$f_2 \cdot \dots \cdot f_n(\tau) = 1.$$

*Then  $s \equiv t$  modulo 2.*

**Proof.** By the definition of the product of a 0-cochain with an  $(n - 1)$ -cochain

$$f_2 \cdot \dots \cdot f_n(\tau) = 1$$

if and only if  $h_1(\lambda^0 \tau) = 1$  and  $f_2 \cdot \dots \cdot f_n(\tau) = 1$ . It follows that

$$\sum_{\tau} h_1 \cdot f_2 \cdot \dots \cdot f_n(\tau) = t \text{ modulo } 2,$$

where the sum is taken over all  $(n - 1)$ -cubes  $\tau \subset \text{bd } K$ . Also, obviously,

$$\sum_{\sigma} f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = s \text{ modulo } 2,$$

where the sum is taken over all  $n$ -cubes  $\sigma$  of  $K$ . Therefore, it is sufficient to prove that the two sums above are equal. In order to prove this, note that the identity  $\partial^* \circ \partial^* = 0$  implies that  $\partial^* f_i = 0$  for every  $i$ . Together with Lemma 7.2 this implies that

$$\partial^*(h_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n) = f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n.$$

Now Lemma 7.3 implies that these two sums are indeed equal. ■

**7.5. Corollary.** Let  $h_1, h_2, \dots, h_n$  be 0-cochains of  $K$  and let  $f_i = \partial^* h_i$  for every  $i \in I$ . Let  $s$  be the same number as in Theorem 7.4. Suppose that

$$h_i(\rho) = 1 \text{ if } \rho \subset \mathcal{A}_i \text{ and}$$

$$h_i(\rho) = 0 \text{ if } \rho \subset \mathcal{B}_i$$

for every  $i \in I$  and let  $t_1$  be the number of  $(n - 1)$ -cubes  $\tau \subset \mathcal{A}_1$  such that

$$f_2 \cdot \dots \cdot f_n(\tau) = 1.$$

Then  $s \equiv t_1 \text{ modulo } 2$ .

**Proof.** Let  $t$  be the number defined in Theorem 7.4. It is sufficient to prove that  $t_1 = t$ . Let us consider an  $(n - 1)$ -cube  $\tau \subset \text{bd } K$ . If  $\tau \subset \mathcal{B}_1$ , then  $h_1(\lambda^0 \tau) = 0$  and hence  $\tau$  is not among the cubes counted in Theorem 7.4. If

$$\tau \subset \mathcal{A}_i \text{ or } \tau \subset \mathcal{B}_i,$$

then  $f_i(\varepsilon) = 0$  for every 1-face  $\varepsilon$  of  $\tau$ . If  $i \geq 2$ , this implies that  $f_2 \cdot \dots \cdot f_n(\tau) = 0$  and hence  $\tau$  is again not among the cubes counted in Theorem 7.4. It follows that  $t_1 = t$  and hence the corollary follows from Theorem 7.4. ■

**7.6. Theorem.** *Under the assumptions of Corollary 7.5 the number  $s$  of  $n$ -cubes  $\sigma$  such that  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$  is odd.*

**Proof.** Suppose that  $n = 1$ . Then we can identify  $h_1$  with a sequence of 0's and 1's starting with 1 and ending with 0. The total number of changes from 0 to 1 and from 1 to 0 in such a sequence is odd. But the number of changes is equal to  $s$ . This proves the theorem for  $n = 1$ . The assumptions of Corollary 7.5 imply that the same assumptions hold for  $n - 1$  in the role of  $n$ , the face  $\mathcal{A}_1$  in the role of  $K$ , and the restrictions of  $h_2, h_3, \dots, h_n$  to  $\mathcal{A}_1$  in the role of  $h_1, h_2, \dots, h_n$ . Therefore an induction by  $n$  completes the proof. ■

**Another proof.** For every  $i \in I$  let us define a 0-cochain  $u_i$  as follows:  $u_i(\rho) = 1$  if the 0-cube  $\rho$  is contained in  $\mathcal{A}_i$  and  $u_i(\rho) = 0$  otherwise. Clearly, the assumptions of the Corollary 7.5 hold for  $h_i = u_i$ . For every  $i \in I$ , let  $g_i = \partial^* u_i$ . Then  $g_i(\varepsilon) = 1$  if and only if the root of the 1-cube  $\varepsilon$  belongs to  $\mathcal{A}_i$  and its direction is  $\{i\}$ . Hence the summand

$$g_1(\tau_1) \cdot g_2(\tau_2) \cdot \dots \cdot g_n(\tau_n)$$

of  $g_1 \cdot g_2 \cdot \dots \cdot g_n(\sigma)$  for some  $n$ -cube  $\sigma$  is equal to 1 if and only if for every  $i \in I$  the root of the 1-cube  $\tau_i$  belongs to  $\mathcal{A}_i$  and the direction of  $\tau_i$  is  $\{i\}$ .

We claim that there is only one  $n$ -cube  $\sigma$  and only one sequence  $\tau_1, \tau_2, \dots, \tau_n$  with these properties. Let  $v_0, v_1, \dots, v_n$  be the corresponding pivot sequence. Then the  $i$ th coordinate of  $v_{i-1}$  is equal to 0 because  $v_{i-1}$  is the root of  $\tau_i$  and hence  $v_{i-1} \in \mathcal{A}_i$ . Similarly, the  $i$ th coordinate of  $v_i$  is equal to 1 because  $v_i$  is the peak of  $\tau_i$ . Since no coordinate decreases along a pivot sequence, this implies that the  $i$ th coordinate of  $v_j$  is equal to 0 if  $j \leq i - 1$  and is equal to 1 if  $j \geq i$ . It follows that  $v_j = (1, \dots, 1, 0, \dots, 0)$ , where  $j$  coordinates equal to 1 are followed by  $n - j$  coordinates equal to 0. This proves our claim. Clearly, this claim implies that

$$\sum_{\sigma} g_1 \cdot g_2 \cdot \dots \cdot g_n(\sigma) = 1,$$

where  $\sigma$  runs over all  $n$ -cubes. Therefore the theorem is true for  $h_i = u_i$ . Let us compare the general case with this one. Let  $d_i = h_i - u_i$ . Then  $\partial^* d_i = f_i - g_i$  for every  $i \in I$ . Together with Lemma 7.2 and the identity  $\partial^* \circ \partial^* = 0$  this implies that

$$\partial^*(d_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n) = f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n - g_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n,$$

$$\partial^*(g_1 \cdot d_2 \cdot f_3 \cdot \dots \cdot f_n) = g_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n - g_1 \cdot g_2 \cdot f_3 \cdot \dots \cdot f_n,$$

.....

$$\partial^*(g_1 \cdot g_2 \cdot \dots \cdot g_{n-1} \cdot d_n) = g_1 \cdot g_2 \cdot \dots \cdot g_{n-1} \cdot f_n - g_1 \cdot g_2 \cdot \dots \cdot g_{n-1} \cdot g_n.$$

By summing these equalities we see that

$$(19) \quad \partial^* \Sigma = f_1 \cdot f_2 \cdot \dots \cdot f_n - g_1 \cdot g_2 \cdot \dots \cdot g_n,$$

where

$$\Sigma = d_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_n + g_1 \cdot d_2 \cdot f_3 \cdot \dots \cdot f_n + \dots + g_1 \cdot g_2 \cdot \dots \cdot g_{n-1} \cdot d_n.$$

We claim that every summand of  $\Sigma$  is equal to 0 on  $(n-1)$ -cubes contained in  $\text{bd} K$ . Let us consider the  $j$ th summand, where  $j \in I$ . It is equal to  $g \cdot d \cdot f$ , where

$$g = g_1 \cdot \dots \cdot g_{j-1}, \quad d = d_j, \quad \text{and} \quad f = f_{j+1} \cdot \dots \cdot f_n.$$

Let  $\tau$  be an  $(n-1)$ -cube. Then

$$(20) \quad g \cdot d \cdot f(\tau) = \sum g(\tau_1) \cdot d(\tau_2) \cdot f(\tau_3)$$

where the sum is taken over the triples  $\tau_1, \tau_2, \tau_3$  subject, up to the notations, to the same conditions as the sequences  $\tau_1, \tau_2, \dots, \tau_r$  in (18).

Arguing as in the case of the product  $g_1 \cdot g_2 \cdot \dots \cdot g_n$ , we see that if  $g(\tau_1) \neq 0$ , then the direction of  $\tau_1$  is  $\{1, 2, \dots, j-1\}$  and the peak of  $\tau_1$  has the form  $\{v\}$ , where

$$v = (1, \dots, 1, a_j, \dots, a_n)$$

with the first  $j-1$  coordinates equal to 1. Since  $v \in \tau$ , it follows that in this case  $\tau$  is not contained in any face  $\mathcal{A}_i, \mathcal{B}_i$  with  $i \leq j-1$ .

The peak  $\{v\}$  of  $\tau_1$  is also the root of the 0-cube  $\tau_2$ . If  $d(\tau_2) \neq 0$ , then  $0 < a_j < k$ . Similarly to the previous paragraph, this implies that  $\tau$  is not contained in the faces  $\mathcal{A}_j, \mathcal{B}_j$ .

Finally, arguing as in the proof of Corollary 7.5, we see that if  $f(\tau_3) \neq 0$ , then  $\tau_3$  and hence  $\tau$  is not contained in any face  $\mathcal{A}_i, \mathcal{B}_i$  with  $i \geq j+1$ .

By collecting all this information together we see that if at least one of terms of sum (20) is  $\neq 0$ , then  $\tau$  is not contained in any  $(n-1)$ -face of  $K$ , i.e. not contained in the boundary  $\text{bd} K$ . Since this is true for every  $j \in I$ , it follows that  $\Sigma(\tau) = 0$  for every  $(n-1)$ -cube  $\tau$  contained in  $\text{bd} K$ . Now Lemma 7.3 implies that

$$\sum_{\sigma} \partial^* \Sigma(\sigma) = 0,$$

where the sum is taken over all  $n$ -cubes  $\sigma$  of  $K$ .

Together with (19) this implies that

$$\sum_{\sigma} f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = \sum_{\sigma} g_1 \cdot g_2 \cdot \dots \cdot g_n(\sigma) = 1.$$

The theorem follows. ■

**Remark.** The second proof is longer, but more straightforward. While the first proof compares the situation with a similar one in lower dimension, the second proof compares the general situation with the simplest one in the same dimension and avoids induction by  $n$ . Only the second method is available in Lusternik–Schnirelmann situation. See Section 8.

**7.7. Theorem.** Suppose that  $c_1, c_2, \dots, c_n$  are subsets of  $K$  and let  $\bar{c}_i = K \setminus c_i$ . If

$$\mathcal{A}_i \subset c_i \text{ and } \mathcal{B}_i \subset \bar{c}_i$$

for every  $i \in I$ , then there is a pivot sequence  $v_0, v_1, \dots, v_n$  such that

$$\{v_{i-1}, v_i\} \text{ intersects both } c_i \text{ and } \bar{c}_i$$

for every  $i \in I$ . In particular, there is an  $n$ -cube intersecting all sets  $c_i$  and  $\bar{c}_i$ ,  $i \in I$ . The number of such pivot sequences is odd.

**Proof.** For every  $i \in I$  let us define a 0-cochain  $h_i$  as follows:  $h_i(\rho) = 1$  if the 0-cube  $\rho$  is contained in  $c_i$  and  $h_i(\rho) = 0$  otherwise. Let  $f_i = \partial^* h_i$  for every  $i \in I$ . Then  $f_i(\varepsilon) = 1$  if the 1-cube  $\varepsilon$  intersects both  $c_i$  and  $\bar{c}_i$ , and  $f_i(\varepsilon) = 0$  otherwise. It follows that for every  $n$ -cube  $\sigma$

$$f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma)$$

is the image in  $\mathbb{F}_2$  of the number of pivot sequences  $v_0, v_1, \dots, v_n \in \sigma$  such that the 1-cube  $\{v_{i-1}, v_i\}$  intersects both  $c_i$  and  $\bar{c}_i$  for every  $i \in I$ . It follows that the sum

$$\sum_{\sigma} f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma),$$

where  $\sigma$  runs over all  $n$ -cubes, is equal to the number of pivot sequences with required properties taken modulo 2. Therefore it is sufficient to prove that this sum is equal to 1. Clearly, this sum is equal to the number  $s$  of  $n$ -cubes  $\sigma$  such that

$$f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1.$$

taken modulo 2. The assumptions about the subsets  $c_i$  are equivalent to the assumptions of Theorem 7.6 about the cochains  $h_i$ . Therefore Theorem 7.6 implies that  $s$  is odd. ■

**7.8. Theorem.** Let  $D_1, D_2, \dots, D_n$  be closed subsets of  $[0, 1]^n$  such that for every  $i$  the set  $D_i$  contains  $A_i$  and is disjoint from  $B_i$ . Let  $\bar{D}_i$  be the closure of  $[0, 1]^n \setminus D_i$ . Then

$$(D_1 \cap D_2 \cap \dots \cap D_n) \cap (\bar{D}_1 \cap \bar{D}_2 \cap \dots \cap \bar{D}_n) \neq \emptyset.$$

**Proof.** The sets  $D_1, D_2, \dots, D_n$  together with the sets  $\bar{D}_1, \bar{D}_2, \dots, \bar{D}_n$  form a closed covering of  $[0, 1]^n$  (of course, every pair  $D_i, \bar{D}_i$  already forms a covering). Let  $\varepsilon > 0$  be a Lebesgue number of this covering. Suppose that the number  $k$  is chosen to be so large that an  $n$ -dimensional cube in  $\mathbb{R}^n$  with the sides of the length  $1/k$  has diameter  $< \varepsilon$ . Let  $p: K \rightarrow [0, 1]^n$  be the map defined by

$$p(a_1, a_2, \dots, a_n) = (a_1/k, a_2/k, \dots, a_n/k).$$

For each  $i \in I$  let  $c_i = p^{-1}(D_i)$  and  $\bar{c}_i = K \setminus c_i$ . Then

$$\mathcal{A}_i \subset c_i \quad \text{and} \quad \mathcal{B}_i \subset \bar{c}_i$$

for every  $i \in I$ . Therefore, the assumptions of Theorem 7.7 hold and this theorem implies that there exists an  $n$ -cube  $\sigma$  of  $K$  intersecting all sets  $c_i, \bar{c}_i, i \in I$ . Since, obviously,

$$p(c_i) \subset D_i \quad \text{and} \quad p(\bar{c}_i) \subset \bar{D}_i$$

for every  $i \in I$ , the image  $p(\sigma)$  intersects all sets  $D_i$  and  $\bar{D}_i, i \in I$ . The image  $p(\sigma)$  is equal to the set of vertices of an  $n$ -dimensional cube with the sides of the length  $1/k$ . By the choice of  $k$  this implies that the diameter of  $p(\sigma)$  is  $< \varepsilon$ . It follows that every point of  $p(\sigma)$  is at the distance  $< \varepsilon$  from each of the sets  $D_i$  and  $\bar{D}_i, i \in I$ . By the choice of  $\varepsilon$  this implies that the intersection of all these sets is non-empty. ■

**Theorem about partitions.** Let  $C_1, C_2, \dots, C_n$  be closed subsets of  $[0, 1]^n$  such that for every  $i$  the set  $C_i$  is a partition between  $A_i$  and  $B_i$  in the sense that  $A_i$  and  $B_i$  are contained in different components of the complement  $[0, 1]^n \setminus C_i$ . Then  $C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset$ .

**Proof.** This theorem was already proved in Section 1. Now we would like to deduce it from Theorem 7.7 using Theorem 7.8 as an intermediary. Let  $U_i$  be the component of  $[0, 1]^n \setminus C_i$  containing  $A_i$ , and let  $D_i$  be its closure. Let  $\bar{D}_i$  be the closure of  $[0, 1]^n \setminus D_i$ . Then  $B_i$  is contained in  $\bar{D}_i$ . If a point  $x \in [0, 1]^n$  is not contained in  $C_i$ , then either  $x \in U_i$ , or  $x$  belongs to some other component of the complement  $[0, 1]^n \setminus C_i$ . In the first case  $x \notin \bar{D}_i$ , in the second case  $x \notin D_i$ . It follows that

$$D_i \cap \bar{D}_i \subset C_i.$$

It remains to apply Theorem 7.8. ■

**Lebesgue first covering theorem revisited.** Theorem about partitions seems to be weaker than Lebesgue first covering theorem. But the latter can be deduced from Theorem about partitions by using Lebesgue fusion of sets (see Section 1) and some elementary topology.

**7.9. Lemma.** *Let  $Z \subset [0, 1]^n$  be a closed set and  $E, F \subset Z$  be its closed subsets such that  $E \cup F = Z$ . If the sets  $E, F$  are disjoint from  $B_i, A_i$  respectively, then there exists a closed set  $C_i \subset [0, 1]^n$  such that  $C_i \cap Z = E \cap F$  and  $C_i$  is separating  $A_i$  from  $B_i$ .*

**Proof.** Clearly,  $E \cap A_i$  and  $E \cap F$  are disjoint closed subsets of  $E$ . It follows that there exists a continuous function  $\varphi: E \rightarrow [-1, 0]$  such that

$$\varphi^{-1}(-1) = E \cap A_i \quad \text{and} \quad \varphi^{-1}(0) = E \cap F.$$

Similarly, there exists a continuous function  $\psi: F \rightarrow [0, 1]$  such that

$$\psi^{-1}(0) = E \cap F \quad \text{and} \quad \psi^{-1}(1) = F \cap B_i.$$

Clearly, there is a continuous function  $Z \rightarrow [-1, 1]$  equal to  $\varphi$  on  $E$  and to  $\psi$  on  $F$ . Moreover, this function can be extended to a function  $A_i \cup Z \cup B_i \rightarrow [-1, 1]$  equal to  $-1$  on  $A_i$  and to  $1$  on  $B_i$ . Finally, the latter function can be extended to a continuous function  $\rho: [0, 1]^n \rightarrow [-1, 1]$ . A trivial verification shows that one can take  $C_i = \rho^{-1}(0)$ . ■

**7.10. Lemma.** *Let  $Z_0 \supset Z_1 \supset \dots \supset Z_n$  be a decreasing sequence of sets. Suppose that  $C_1, C_2, \dots, C_n$  are subsets of  $Z_0$  such that  $C_m \cap Z_{m-1} = Z_m$  for all  $m = 1, 2, \dots, n$ . Then  $C_1 \cap C_2 \cap \dots \cap C_m = Z_m$  for all  $m = 1, 2, \dots, n$ . ■*

**Lebesgue first covering theorem.** *Let  $D_1, D_2, \dots, D_r$  be a covering of the unit cube  $[0, 1]^n$  by closed sets. Suppose that none of the sets  $D_i$  intersects two opposite  $(n-1)$ -faces of  $[0, 1]^n$ . Then among the sets  $D_i$  there are  $n+1$  sets with non-empty intersection.*

**Proof.** Now we will deduce this theorem from Theorem about partitions. Lebesgue fusion of sets construction from Section 1 leads to a covering of  $[0, 1]^n$  by sets  $E_1, E_2, \dots, E_{n+1}$  such that these sets are unions of disjoint collections of the sets  $D_i$  and the conditions (i) and (ii) of Theorem 1.5 hold (for  $E_i$  in the role of  $e_i$ ). In particular,  $E_i$  is disjoint from  $A_j$  if  $i > j$  and is disjoint from  $B_i$  if  $i \leq n$ . For  $m \in I$  let

$$X_m = E_1 \cap E_2 \cap \dots \cap E_m,$$

$$Y_m = E_{m+1} \cup E_{m+2} \cup \dots \cup E_{n+1}.$$

Then  $X_m$  is disjoint from  $B_m$  and  $Y_m$  is disjoint from  $A_m$  for every  $m \in I$ .

Let  $Z_0 = [0, 1]^n$ . Obviously,  $Z_0 = X_1 \cup Y_1$ . The conditions (i) and (ii) imply that  $X_1$  contains  $A_1$  and is disjoint from  $B_1$ , and  $Y_1$  contains  $B_1$  and is disjoint from  $A_1$ . Hence

$$A_1 \subset X_1 \setminus (X_1 \cap Y_1) \quad \text{and} \quad B_1 \subset Y_1 \setminus (X_1 \cap Y_1).$$

Obviously, the sets

$$X_1 \setminus (X_1 \cap Y_1) \quad \text{and} \quad Y_1 \setminus (X_1 \cap Y_1)$$

are closed in  $Z_0 \setminus (X_1 \cap Y_1)$ . It follows that  $X_1 \cap Y_1$  separates  $A_1$  from  $B_1$  in  $Z_0$ .

Suppose that  $1 \leq m \leq n$  and let  $Z_m = X_m \cap Y_m$ . In particular,  $Z_1 = X_1 \cap Y_1$ . The sets  $Z_m$  are analogues of the cubical sets  $|\gamma_m|$  from the proof of Theorem 1.5. Similarly to that proof, we would like to prove that  $Z_n \neq \emptyset$ . In order to do this, we will “extend” each set  $Z_m$  to a set  $C_m$  separating  $A_m$  and  $B_m$  in  $[0, 1]^n$ . Since the set  $Z_1$  itself has this property, we can set  $C_1 = Z_1$ .

To begin with, let us note that  $Y_{m-1} = E_m \cup Y_m$  and hence

$$\begin{aligned} Z_{m-1} &= X_{m-1} \cap (E_m \cup Y_m) \\ &= (X_{m-1} \cap E_m) \cup (X_{m-1} \cap Y_m) \\ &= X_m \cup (X_{m-1} \cap Y_m). \end{aligned}$$

Clearly,  $X_{m-1} \cap Y_m$  is disjoint from  $A_m$  together with  $Y_m$ . By applying Lemma 7.9 to

$$Z = Z_{m-1}, \quad E = X_m, \quad \text{and} \quad F = X_{m-1} \cap Y_m$$

we see that there exists a closed set  $C_m$  separating  $A_m$  from  $B_m$  and such that

$$C_m \cap Z_{m-1} = X_m \cap X_{m-1} \cap Y_m = Z_m.$$

Therefore  $C_m \cap Z_{m-1} = Z_m$  for every  $m \in I$ . Since, obviously,  $Z_0 \supset Z_1 \supset \dots \supset Z_n$ , Lemma 7.10 implies that  $C_1 \cap C_2 \cap \dots \cap C_n = Z_n$ . Theorem about partitions implies that the intersection  $C_1 \cap C_2 \cap \dots \cap C_n$  is non-empty. It follows that  $Z_n$  is non-empty. But

$$Z_n = E_1 \cap E_2 \cap \dots \cap E_{n+1}$$

and hence the intersection of the sets  $E_1, E_2, \dots, E_{n+1}$  is non-empty. Since these sets are unions of disjoint collections of the sets  $D_1, D_2, \dots, D_r$ , it follows that among the sets  $D_1, D_2, \dots, D_r$  there are  $n + 1$  sets with non-empty intersection. ■



## 8. The discrete sphere and products of cubical cochains

**The discrete sphere.** Since we are going to explore the properties of the central symmetry of an  $n$ -dimensional sphere, it is convenient to replace the cube  $K$  from Section 7 by a centrally symmetric  $(n + 1)$ -dimensional cube with the center at  $0 \in \mathbb{R}^{n+1}$ . Let  $I$  be the set  $\{1, 2, \dots, n + 1\}$  and let  $k$  be a natural number. Our symmetric cube is

$$C = \{-k, \dots, -1, 0, 1, \dots, k\}^{n+1}.$$

The *discrete sphere*  $S$  of dimension  $n - 1$  and size  $2k$  is the set of all points

$$(a_1, a_2, \dots, a_{n+1}) \in C$$

such that  $a_i = -k$  or  $a_i = k$  for some  $i \in I$ . All notions and results from Section 7, except of products, have obvious analogues for the symmetric cube  $C$ . In particular, the boundary  $\text{bd}C$  is defined, and, obviously,  $S = \text{bd}C$ . The  $m$ -cubes of  $S$  are defined as  $m$ -cubes of  $C$  contained in  $S$ . The  $m$ -chains of  $S$  are the formal sums of  $m$ -cubes of  $S$  with coefficients in  $\mathbb{F}_2$ , and the  $m$ -cochains are the  $\mathbb{F}_2$ -valued functions on the set of  $m$ -cubes of  $S$ . The boundary operator  $\partial$  on chains and the coboundary operator  $\partial^*$  on cochains are defined as before, but taking only the cubes of  $S$  into account.

**Symmetric and asymmetric cochains.** Recall that  $\iota: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  is the antipodal involution,  $\iota(x) = -x$ . Obviously,  $\iota$  leaves the sphere  $S$  invariant and maps every  $m$ -cube of  $S$  into an  $m$ -cube of  $S$ . This leads to a self-map  $\iota_*$  of the  $\mathbb{F}_2$ -vector space of  $m$ -chains of  $S$ . The dual self-map of the  $\mathbb{F}_2$ -vector space of  $m$ -cochains of  $S$  is denoted by  $\iota^*$ . If  $f$  is an  $m$ -cochain considered as an  $\mathbb{F}_2$ -valued functions on the set of  $m$ -cubes, then

$$\iota^*(f)(\sigma) = f(\iota_*(\sigma))$$

for every  $m$ -cube  $\sigma$  of  $S$ . Obviously, a cube  $\tau$  is a face of another cube  $\sigma$  if and only if  $\iota(\tau)$  is a face of  $\iota(\sigma)$ . It follows that  $\partial$  and  $\partial^*$  commute with  $\iota_*$  and  $\iota^*$  respectively, i.e.

$$\partial \circ \iota_* = \iota_* \circ \partial \quad \text{and} \quad \partial^* \circ \iota^* = \iota^* \circ \partial^*.$$

An  $m$ -cochain  $f$  is said to be *symmetric* if  $\iota^*(f) = f$ , i.e. if

$$f(\iota_*(\sigma)) = f(\sigma)$$

for every  $m$ -cube  $\sigma$  of  $S$ . If  $f$  is symmetric, then  $\partial^*(f)$  is also symmetric.

Let us denote by  $\iota$  also the map  $\mathbb{F}_2 \rightarrow \mathbb{F}_2$  defined as follows:  $\iota(0) = 1$  and  $\iota(1) = 0$ . Clearly,  $\iota \circ \iota$  is the identity map, i.e.  $\iota: \mathbb{F}_2 \rightarrow \mathbb{F}_2$  is an involution. But  $\iota$  is not an auto-

morphism of  $\mathbb{F}_2$ . An  $m$ -cochain  $f$  is said to be *asymmetric* if  $\iota^*(f) = \iota \circ f$ , i.e. if

$$f(\iota_*(\sigma)) = \iota(f(\sigma))$$

for every  $m$ -simplex  $\sigma$  of  $S$ . This property is not preserved by the coboundary operator  $\partial^*$ .

**8.1. Lemma.** *If  $h$  is an asymmetric 0-cochain, then the cochain  $\partial^*h$  is symmetric.*

**Proof.** Let  $\sigma = \{u, v\}$  be a 1-cube of  $S$ . Then

$$\partial^*h(\sigma) = h(u) + h(v)$$

and hence  $\partial^*h(\sigma) = 1$  if and only if one of the values  $h(u)$ ,  $h(v)$  is equal to 1 and the other to 0. Equivalently,

$$\partial^*h(\sigma) = 1 \quad \text{if and only if} \quad h(u) \neq h(v).$$

Since  $h$  is asymmetric, the latter condition is equivalent to

$$h(\iota(u)) \neq h(\iota(v))$$

and hence to  $\partial^*h(\iota_*(\sigma)) = 1$ . It follows that  $\iota^*(\partial^*h) = \partial^*h$ . ■

**8.2. Lemma.** *If  $f, g$  are asymmetric  $m$ -cochains, then the  $m$ -cochain  $f + g$  is symmetric.*

**Proof.** Let  $\sigma$  be an  $m$ -cube of  $S$ . Then

$$f(\sigma) + g(\sigma) = 1$$

if and only if one of the values  $f(\sigma)$ ,  $g(\sigma)$  is equal to 1 and the other to 0, i.e. if and only if  $f(\sigma) \neq g(\sigma)$ . Since  $f, g$  are asymmetric, the latter condition is equivalent to

$$f(\iota_*(\sigma)) \neq g(\iota_*(\sigma))$$

and hence to  $\iota^*(f)(\sigma) + \iota^*(g)(\sigma) = 1$ . It follows that  $\iota^*(f) + \iota^*(g) = f + g$ . ■

**Summing values of symmetric  $n$ -cochains.** For a symmetric  $n$ -cochain  $f$  we will interpret

$$\sum_{\sigma} f(\sigma)$$

as the sum over any set of  $n$ -cubes  $\sigma$  containing one element of every pair of the form  $\rho$ ,  $\iota_*(\rho)$ . Clearly, this sum does not depend on the choice of such set.

**8.3. Lemma.** *If  $f$  is a symmetric  $(n-1)$ -cochain, then  $\sum_{\sigma} \partial^* f(\sigma) = 0$ .*

**Proof.** It is similar to the proof of Lemma 7.3. If one considers  $(n-1)$ -cochains as formal sums of  $(n-1)$ -cubes, it is sufficient to prove the lemma in the case when  $f = \tau + \iota_*(\tau)$  for an  $(n-1)$ -cube  $\tau$ . The  $(n-1)$ -cube  $\tau$  is a face of exactly two  $n$ -cubes, say, the  $n$ -cubes  $\sigma_1$  and  $\sigma_2$ . Then  $\partial^* f(\sigma)$  is 1 if  $\sigma = \sigma_1, \sigma_2, \iota_*(\sigma_1), \iota_*(\sigma_2)$ , and is 0 otherwise. Obviously,  $\iota_*(\sigma_1) \neq \sigma_2$  and hence we may assume that  $\sigma$  in the sum runs over a set containing  $\sigma_1, \sigma_2$ . Clearly, in this case the sum is equal to  $1 + 1 = 0 \in \mathbb{F}_2$ . ■

**Products of cochains of  $C$  and  $S$ .** The most obvious analogue of the products from Section 7 results from identifying  $C$  with  $\{0, 1, \dots, 2k\}^{n+1}$  using the translation by  $(k, k, \dots, k)$ . Unfortunately, this leads to a product which is not  $\iota$ -invariant. More precisely, if the product is defined in this way, then  $\iota^*(g \cdot h) = \iota^*(h) \cdot \iota^*(g)$ , i.e.  $\iota^*$  is not a homomorphism, but an anti-homomorphism (the product of cochains is not commutative).

Let  $\|\cdot\|_1$  be the  $l_1$ -norm on  $\mathbb{R}^{n+1}$ . Recall that if  $x = (x_1, x_2, \dots, x_{n+1})$ , then

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_{n+1}|.$$

Let  $\sigma$  be a cube of  $S$  or  $C$ . Let us define the *root*  $\lambda^0 \sigma$  of  $\sigma$  as the 0-face  $\lambda^0 \sigma = \{r\}$  of  $\sigma$  such that the  $l_1$ -norm  $\|r\|_1$  is minimal among  $l_1$ -norms of elements of  $\sigma$ . Similarly, the *peak*  $\lambda^1 \sigma$  of  $\sigma$  is defined as the 0-face  $\lambda^1 \sigma = \{p\}$  of  $\sigma$  such that  $\|p\|_1$  is maximal among  $l_1$ -norms of elements of  $\sigma$ . More generally, every cube  $\sigma$  of  $C$  has the form

$$\sigma = \prod_{i=1}^{n+1} \rho_i,$$

where for every  $i$  either  $\rho_i = \{a_i, b_i\}$  for some integers  $a_i, b_i$  between  $-k$  and  $k$  such that  $|a_i - b_i| = 1$ , or  $\rho_i = \{a_i\}$  for some integer  $a_i$  between  $-k$  and  $k$ . We may assume that in the first case  $|a_i| < |b_i|$ . The *direction*  $A(\sigma)$  of  $\sigma$  is the set of subscripts  $i$  such that  $\rho_i$  consists of two elements. Suppose that  $H \subset A(\sigma)$  and  $\varepsilon = 0$  or  $1$ . Let us replace in the above product every factor  $\rho_i = \{a_i, b_i\}$  with  $i \in H$  by  $\{a_i\}$  if  $\varepsilon = 0$  and by  $\{b_i\}$  if  $\varepsilon = 1$ . The resulting product is denoted by

$$\lambda_H^\varepsilon \sigma.$$

Clearly, if  $A = A(\sigma)$ , then  $\lambda^0 \sigma = \lambda_A^0 \sigma$  and  $\lambda^1 \sigma = \lambda_A^1 \sigma$ .

The key advantage of these definitions is their  $\iota$ -invariance. In particular, if  $\tau$  is the root or peak of  $\sigma$ , then, obviously,  $\iota_*(\tau)$  is the root or peak of  $\iota_*(\sigma)$  respectively (in contrast with the definition based on the above translation, which leads to interchanging roots and peaks by  $\iota_*$ ). More generally, if  $H \subset A(\sigma)$ , then

$$\lambda_H^\varepsilon (\iota_*(\sigma)) = \iota_*(\lambda_H^\varepsilon \sigma).$$

Now we can define the product of cochains by one of the formulas (14) or (17) from Section 7. The resulting definition is  $\iota$ -invariant, i.e. if  $f, g$  are cochains of  $S$  or  $C$ , then

$$\iota^*(f \cdot g) = \iota^*(g) \cdot \iota^*(f).$$

In particular, the product of two or more symmetric cochains of  $S$  is symmetric. As in Section 7, the products of cochains on  $C$  and  $S$  are associative and satisfy Leibniz formula (i.e. analogues of Lemmas 7.1 and 7.2 hold). The proofs are the same.

**Products of 1-cochains.** If  $f_1, f_2, \dots, f_n$  are 1-cochains and  $\sigma$  is an  $n$ -cube of  $S$ , then

$$f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = \sum f_1(\tau_1) \cdot f_2(\tau_2) \cdot \dots \cdot f_n(\tau_n),$$

where the sum is taken over all sequences  $\tau_1, \tau_2, \dots, \tau_n$  such that  $\tau_i$  is a 1-face of  $\sigma$ , the directions of these faces are pair-wise disjoint, the root of  $\tau_1$  is equal to the root of  $\sigma$ , and the peak of  $\tau_i$  is equal to the root of  $\tau_{i+1}$  for all  $i \leq n-1$ . These conditions imply that the peak of  $\tau_n$  is equal to the peak of  $\sigma$ . Every such sequence  $\tau_1, \tau_2, \dots, \tau_n$  has the form

$$\tau_i = \{v_{i-1}, v_i\}$$

for a sequence  $v_0, v_1, \dots, v_n \in \sigma$  such that the 0-cube  $\{v_0\}$  is the root of  $\sigma$ , the 0-cube  $\{v_n\}$  is the peak of  $\sigma$ , and passing from  $v_{i-1}$  to  $v_i$  increases the *absolute value* of one of the coordinates by 1 (the coordinate itself may decrease by 1) and leaves other coordinates unchanged. We will call such sequences  $v_0, v_1, \dots, v_n$  *pivot sequences*.

**8.4. Theorem.** *Let  $h_1, h_2, \dots, h_n$  be asymmetric 0-cochains of  $S$  and let  $f_i = \partial^* h_i$  for every  $i = 1, 2, \dots, n$ . Then there exists an  $n$ -cube  $\sigma$  of  $S$  such that*

$$(21) \quad f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1.$$

*Moreover, the set of such  $n$ -cubes is invariant under the involution  $\iota$  and consists of an odd number of pairs of  $n$ -cubes of the form  $\rho, \iota_*(\rho)$ .*

**Proof.** By Lemma 8.1 every cochain  $f_i$  is symmetric. This implies that the product cochain  $f_1 \cdot f_2 \cdot \dots \cdot f_n$  is symmetric. In turn, this implies that the set of  $n$ -cubes  $\sigma$  satisfying (21) is equal to the union of several disjoint pairs of the form  $\rho, \iota_*(\rho)$ . The number of such pairs is equal modulo 2 to

$$(22) \quad \sum_{\sigma} f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma),$$

where the sum is understood as was explained before Lemma 8.3. Our plan is to prove that (22) does not depend on the choice of asymmetric 0-cochains  $h_i$  and then compute this sum for a particular choice of these cochains, as in the second proof of Theorem 7.7.

Let us look what happens if  $h_1$  is replaced by some asymmetric 0-chain  $g$ . Let  $f = \partial^* g$  and  $d_1 = h_1 - g = h_1 + g$ . Lemma 8.2 implies that  $d_1$  is a symmetric 0-chain. Clearly,

$$f_1 - f = \partial^* d_1.$$

Since  $f_i = \partial^* h_i$  for every  $i$ , the identity  $\partial^* \circ \partial^* = 0$  implies that  $\partial^* f_i = 0$  for every  $i$ . Together with Leibniz formula this implies that

$$\partial^* (d_1 \cdot f_2 \cdot \dots \cdot f_n) = f_1 \cdot f_2 \cdot \dots \cdot f_n - f \cdot f_2 \cdot \dots \cdot f_n.$$

Since the  $(n-1)$ -cochain  $d_1 \cdot f_2 \cdot \dots \cdot f_n$  is symmetric, Lemma 8.3 implies that

$$\sum_{\sigma} f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = \sum_{\sigma} f \cdot f_2 \cdot \dots \cdot f_n(\sigma),$$

i.e. replacing  $h_1$  by another asymmetric 0-chain does not change the sum (22). Similar arguments apply to other  $h_i$ , and hence (22) is independent on the choice of cochains  $h_i$ . It remains to compute the sum (22) for a particular choice of cochains  $h_i$ . We will do this in the case when  $h_i = h$  for all  $i$ , where  $h$  is defined as follows. Let  $H$  be the set of points

$$a = (a_1, a_2, \dots, a_{n+1}) \in S$$

such that the first non-zero coordinate  $a_i$  of  $a$  (i.e. the first non-zero number in the sequence  $a_1, a_2, \dots, a_{n+1}$ ) is negative, let  $\bar{H} = S \setminus H$ , and let

$$h(a) = 1 \quad \text{if } a \in H,$$

$$h(a) = 0 \quad \text{if } a \in \bar{H}.$$

Clearly,  $\bar{H} = \iota(H)$  and hence the cochain  $h$  is antisymmetric. Let  $f = \partial^* h$ . Now we are going to compute the  $n$ -fold product  $f \cdot f \cdot \dots \cdot f$ .

Let  $\sigma$  be an  $n$ -cube of  $S$  and  $v_0, v_1, \dots, v_n \in \sigma$  be a pivot sequence. For  $1 \leq i \leq n$  let  $\tau_i = \{v_{i-1}, v_i\}$ . Suppose that

$$f(\tau_1) \cdot f(\tau_2) \cdot \dots \cdot f(\tau_n) = 1.$$

We will show that the  $n$ -cube  $\sigma$  and the sequence  $v_0, v_1, \dots, v_n$  are determined by this condition up to simultaneously replacing them by  $\iota_*(\sigma)$  and  $\iota(v_0), \iota(v_1), \dots, \iota(v_n)$ .

By the definition,  $f(\tau_i) = 1$  if and only if one of the points  $v_{i-1}, v_i$  belongs to  $H$  and the other to  $\bar{H}$ . It follows that the points  $v_0, v_1, \dots, v_n$  alternate between  $H$  and  $\bar{H}$ . It will be convenient to say that a point  $v$  is a point of the type  $H$  if  $v \in H$ , and a point of the type  $\bar{H}$  if  $v \in \bar{H}$ . In such terms, when one moves from  $v_0$  to  $v_n$  along the sequence  $v_0, v_1, \dots, v_n$ , the type has to change at each step and hence has to change  $n$  times.

Let  $v_t = (a_1, a_2, \dots, a_{n+1})$ , and let us treat  $a_1, a_2, \dots, a_{n+1}$  as integer-valued variables depending on  $t = 0, 1, \dots, n$ . Since  $v_0, v_1, \dots, v_n$  is a pivot sequence, each of the coordinates  $a_i$  may change at no more than one step. Since  $\sigma$  is a cube of  $S$ , one of the coordinates does not depend on  $t$  and is equal to either  $-k$  or to  $k$  for all  $t$ . If  $a_i$  is the unchanging coordinate, then the type of  $v_t$  may change no more than  $i - 1$  times. Since the type changes  $n$  times, the unchanging coordinate is  $a_{n+1}$  and it is equal to either  $-k$  or  $k$ . Suppose that  $a_{n+1} = -k$ , the case  $a_{n+1} = k$  being completely similar.

Since  $v_0, v_1, \dots, v_n$  is a pivot sequence, for every  $i$  the absolute value  $|a_i|$  of the coordinate  $a_i$  of  $v_t$  does not decrease when  $t$  increases from 0 to  $n$ . It follows that if  $a_i \neq 0$  for  $t = u$ , then the sign of  $a_i$  remains the same for all  $t \geq u$ . In turn, this implies that the type of  $v_t$  may change no more than  $i - 1$  times in the sequence  $v_u, v_{u+1}, \dots, v_n$  (because there are only  $i - 1$  coordinates available to change the type). On the other hand, the type should change at each step, i.e.  $n - u$  times. Therefore, if  $a_i \neq 0$  for  $t = u$ , then  $i - 1 \geq n - u$ , i.e.  $i \geq n + 1 - u$ .

In particular, if the coordinate  $a_i$  of  $v_0$  is non-zero, then  $i \geq n + 1$ . Therefore the only non-zero coordinate of  $v_0$  is  $a_{n+1} = -k$  and hence  $v_0 = (0, \dots, 0, -k) \in H$ .

If the coordinate  $a_i$  for  $t = 1$  is non-zero, then  $i \geq n$ , and if  $a_n = 0$  for  $t = 1$ , then  $v_1$  and  $v_0$  have the same type, contrary to the assumption. Therefore  $a_n \neq 0$  for  $t = 1$ . Since  $v_0$  has the type  $H$ , the type of  $v_1$  is  $\overline{H}$ . It follows that  $a_n = 1$  for  $t = 1$ , and hence  $v_1 = (0, \dots, 0, 1, -k) \in \overline{H}$ .

If the coordinate  $a_i$  for  $t = 2$  is non-zero, then  $i \geq n - 1$ , and if  $a_{n-1} = 0$  for  $t = 2$ , then  $v_2$  and  $v_1$  have the same type, contrary to the assumption. Therefore  $a_{n-1} \neq 0$  for  $t = 2$ . Since  $v_1$  has the type  $\overline{H}$ , the type of  $v_2$  is  $H$ . It follows that  $a_{n-1} = -1$  for  $t = 2$ , and hence  $v_2 = (0, \dots, 0, -1, 1, -k) \in H$ .

By continuing to argue in this way, we see that the sequence  $v_0, v_1, \dots, v_n$  with the unchanging coordinate  $a_{n+1} = -k$  is indeed uniquely determined. Clearly, there is unique  $n$ -cube  $\sigma$  containing this sequence. Moreover, the sequence  $v_0, v_1, \dots, v_n$  constructed in the course of this argument is a pivot sequence such that the type of  $v_i$  changes at each step. Since there is only one such sequence contained in  $\sigma$ , it follows that

$$f \cdot f \cdot \dots \cdot f(\sigma) = 1.$$

The same is true under the assumption  $a_{n+1} = k$ . Hence there are exactly two  $n$ -cubes  $\sigma$  such that  $f \cdot f \cdot \dots \cdot f(\sigma) = 1$  and  $\iota$  takes each of them to the other, as one can either verify directly, or deduce from the  $\iota$ -invariance of this condition. It follows that

$$\sum_{\sigma} f \cdot f \cdot \dots \cdot f(\sigma) = 1.$$

As we saw, this implies that (22) is equal to 1 for every choice of  $h_1, h_2, \dots, h_n$ . ■

**8.5. Theorem.** Suppose that  $c_1, c_2, \dots, c_n$  are subsets of  $S$  such that  $\iota(c_i) = S \setminus c_i$  for every  $i = 1, 2, \dots, n$ . Then there exists an  $n$ -cube  $\sigma$  of  $S$  such that for every  $i$

$$\sigma \cap c_i \neq \emptyset \quad \text{and} \quad \sigma \cap (S \setminus c_i) \neq \emptyset.$$

**Proof.** This is just a restatement of Theorem 8.4 in terms of subsets of  $S$ . In more details, let  $1 \leq i \leq n$ , and let us define a 0-cochain  $h_i$  as follows:

$$h_i(a) = 1 \quad \text{if} \quad a \in c_i \quad \text{and} \quad h_i(a) = 0 \quad \text{if} \quad a \in S \setminus c_i.$$

Since  $\iota(c_i) = S \setminus c_i$ , every cochain  $h_i$  is asymmetric. Let  $f_i = \partial^* h_i$ . If  $\tau$  is a 1-cube, then  $f_i(\tau) = 1$  if and only if  $\tau$  intersects both  $c_i$  and  $S \setminus c_i$ . Therefore, if

$$f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$$

for an  $n$ -cube  $\sigma$ , then  $\sigma$  intersects both  $c_i$  and  $S \setminus c_i$  for every  $i = 1, 2, \dots, n$ . It remains to apply Theorem 8.4. ■

**8.6. Theorem.** Let  $\mathbb{S}^n$  be the standard unit sphere in  $\mathbb{R}^{n+1}$ . Let  $F_1, F_2, \dots, F_n$  be closed subsets of  $\mathbb{S}^n$ . Suppose that none of them contains a pair of antipodal points. Then the sets  $F_i \cup \iota(F_i)$  do not cover  $\mathbb{S}^n$ .

**Proof.** This is the first Lusternik–Schnirelmann theorem and was proved in Section 5. Now we will deduce it from Theorem 8.5. Let us use the  $l_\infty$ -distance on  $\mathbb{R}^{n+1}$  and  $S$  (see Section 5, for example). Since for every  $i$  the sets  $F_i$  and  $\iota(F_i)$  are compact and disjoint, there is a real number  $\varepsilon > 0$  such that the distance between  $F_i$  and  $\iota(F_i)$  is  $> \varepsilon$  for every  $i$ . Let  $r: S \rightarrow \mathbb{S}^n$  be the restriction of the radial projection  $\mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{S}^n$  to  $S$  and let

$$e_i = r^{-1}(F_i),$$

where  $i = 1, 2, \dots, n$ . Obviously,  $r \circ \iota = \iota \circ r$  and hence  $\iota(e_i) = r^{-1}(\iota(F_i))$ . Clearly,  $e_i$  is disjoint from  $\iota(e_i)$ . Let  $E_i$  be the result of adding to  $e_i$  all points of  $S$  at the  $l_\infty$ -distance 1 from  $e_i$ . Equivalently,  $E_i$  is the union of  $e_i$  and all  $n$ -cubes of  $S$  intersecting  $e_i$ .

If  $k$  is large enough (say, if  $k > 3/\varepsilon$ ), then the distance between the sets  $e_i$  and  $\iota(e_i)$  is  $> 3$  and hence the sets  $E_i$  and  $\iota(E_i)$  are disjoint for every  $i$ . The complement of the union  $E_i \cup \iota(E_i)$  in  $S$  consists of several disjoint pairs of the form  $a, \iota(a)$ . Let  $c_i$  be the result of adding to  $E_i$  one point from each such pair. Then  $\iota(c_i) = S \setminus c_i$ .

By Theorem 8.5 there is an  $n$ -cube  $\sigma$  of  $S$  intersecting  $c_i$  and  $S \setminus c_i$  for every  $i$ . If  $\sigma$  intersects  $e_i$ , then  $\sigma$  is contained in  $E_i \subset c_i$  and hence is disjoint from  $S \setminus c_i$ . Similarly, if  $\sigma$  intersects  $\iota(e_i)$ , then  $\sigma$  is disjoint from  $e_i$ . Therefore  $\sigma$  is disjoint from  $e_i \cup \iota(e_i)$  and hence  $r(\sigma)$  is disjoint from  $F_i \cup \iota(F_i)$  for every  $i$ . The theorem follows. ■

## 9. Kuhn's cubical Sperner lemmas

**In search of the lost cubes: I.** In the introduction to his paper [Ku] H.W. Kuhn wrote

The formal description of the subdivision of a triangle (or, more generally, a simplex) is cumbersome. Does an analogue of the Sperner Lemma holds for the cube, for which subdivision is a trivial formal operation?

The central result of this paper is a combinatorial lemma which is a cubical analogue of Sperner's Lemma ...

Apparently, Kuhn wasn't aware of Lebesgue's work [L1], [L2]. Lebesgue's paper [L2] includes results which could be called "cubical analogues of Sperner's lemma" if not for the fact that they preceded Sperner's work by many years and served as a major motivation for it. Kuhn's "cubical analogue" turned out to be very close to Lebesgue results. It is stated in terms of a partial order  $\leq$  on  $\mathbb{R}^n$  from Section 2. Recall that for two points

$$a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n \quad \text{and} \quad b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$$

$a \leq b$  means that  $a_i \leq b_i$  for every  $i \in I$ . As usual,  $a < b$  means that  $a \leq b$  and  $a \neq b$ . Let  $\mathbb{1} = (1, 1, \dots, 1)$  and  $\mathbb{K} = \{0, 1\}^n$ .

**Kuhn's lemma.** *Let  $L: K \rightarrow \mathbb{K}$  be a map with components  $l_i: K \rightarrow \{0, 1\}$ , where  $i = 1, 2, \dots, n$ . Suppose that*

$$l_i(a_1, a_2, \dots, a_n) = 0 \quad \text{if} \quad a_i = 0 \quad \text{and}$$

$$l_i(a_1, a_2, \dots, a_n) = 1 \quad \text{if} \quad a_i = k$$

*for every  $i \in I$ . Then there is a sequence  $v_0, v_1, \dots, v_n \in K$  such that*

$$(23) \quad v_0 < v_1 < \dots < v_n \leq v_0 + \mathbb{1}$$

*and each map  $l_i$  takes both values 0 and 1 on the set  $\{v_0, v_1, \dots, v_n\}$ .*

**The assumptions and the conclusion of Kuhn's lemma.** It is only natural to introduce the sets  $c_i$  of points  $v \in K$  such that  $l_i(v) = 0$ . Then the complement  $\bar{c}_i = K \setminus c_i$  is the set of points  $v \in K$  such that  $l_i(v) = 1$  and hence  $L$  is determined by the sets  $c_1, c_2, \dots, c_n$ . In terms of these sets the assumptions of Kuhn's lemma mean that  $\mathcal{A}_i \subset c_i$  and  $\mathcal{B}_i \subset \bar{c}_i$  for every  $i$ . The conclusion means that the set  $\{v_0, v_1, \dots, v_n\}$  intersects both  $c_i$  and  $\bar{c}_i$  for every  $i$ . The inequalities  $v_0 \leq v_i \leq v_0 + \mathbb{1}$  mean that  $\{v_0, v_1, \dots, v_n\}$  is contained in an  $n$ -cube of  $K$  and hence (23) means that  $v_0, v_1, \dots, v_n$  is a pivot sequence in the sense of Sections 2 and 7. Now Kuhn's lemma takes the following form.



**An equivalent form of Kuhn's lemma.** Suppose that  $c_1, c_2, \dots, c_n$  are subsets of  $K$  and let  $\bar{c}_i = K \setminus c_i$ . Suppose also that

$$\mathcal{A}_i \subset c_i \quad \text{and} \quad \mathcal{B}_i \subset \bar{c}_i$$

for every  $i \in I$ . Then there is a pivot sequence  $v_0, v_1, \dots, v_n$  such that  $\{v_0, v_1, \dots, v_n\}$  intersects both  $c_i$  and  $\bar{c}_i$  for every  $i \in I$ .

**Kuhn's lemma and products of cochains.** The above form of Kuhn's lemma immediately follows from Theorem 7.7 and is a somewhat weakened form of the latter. In fact, the theory presented in Section 7 was partially motivated by the desire to elucidate Kuhn's lemma.

**Kuhn's reduced labelings.** Kuhn proves his lemma using an induction by  $n$ . As it is often the case, using induction forces to strengthen the statement. Kuhn thinks about the map  $L$  as a labeling of  $K$ , and his strengthening is based on associating to  $L$  a *reduced labeling*  $r: K \rightarrow \{1, 2, \dots, n+1\}$  defined as follows. Let

$$c_{n+1} = K \setminus (c_1 \cup c_2 \cup \dots \cup c_n).$$

Then  $r(v)$  is equal to the minimal integer  $i$  such that  $v \in c_i$ .

**Kuhn's strong lemma.** Let  $L: K \rightarrow \mathbb{K}$  be a map satisfying the assumptions of Kuhn's lemma, and let  $r$  be the reduced labeling associated with  $L$ . Then there exist a pivot sequence  $v_0, v_1, \dots, v_n \in K$  such that the reduced labeling  $r$  maps  $\{v_0, v_1, \dots, v_n\}$  onto the set  $\{1, 2, \dots, n+1\}$ . Moreover, the number of such pivot sequences is odd.

**Triangulating cubes.** Regardless of his interest in working with cubes as opposed to simplices, Kuhn's proofs of his lemmas starts with triangulating cubes, albeit in a canonical way. The canonical triangulations of cubes used by Kuhn were discovered by H. Freudenthal [Fr], who worked with solid cubes. The geometry of these triangulations is more transparent in the case of solid cubes, but in the discrete context it is more natural to work with the corresponding abstract simplicial complexes. We refer to Appendix 1 for a discussion of geometry.

Let us turn  $K$  into an abstract simplicial complex. Naturally, its set of vertices is  $K$  itself. For every pivot sequence  $v_0, v_1, \dots, v_n \in K$  the set  $\{v_0, v_1, \dots, v_n\}$  is a simplex, as also every its subset. There are no other simplices. By an abuse of notation, we denote this simplicial complex also by  $K$ . Also, we denote by  $\text{bd } K$  the simplicial complex having the boundary  $\text{bd } K$  as the set of vertices and the simplices of  $K$  contained in  $\text{bd } K$  as its simplices.

As one may expect, the simplicial complex  $K$  is a pseudo-manifold and its boundary is the simplicial complex  $\text{bd } K$ . By the very definition, it is dimensionally homogenous: every simplex is a face of an  $n$ -simplex. We will not use the fact that  $K$  is strongly connected and leave its proof to the interested readers. Let us prove the non-branching property.

**9.1. Lemma.** *An  $(n - 1)$ -simplex of  $K$  is a face of exactly two  $n$ -simplices of  $K$  if it is not contained in  $\text{bd } K$ , and of exactly one  $n$ -simplex if it is contained in  $\text{bd } K$ .*

**Proof.** Let  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  be the standard basis of the vector space  $\mathbb{R}^n$ .

Let  $v_0, v_1, \dots, v_n$  be a pivot sequence of vertices of  $K$ . Removing a point  $v_i$  from the  $n$ -simplex  $\sigma = \{v_0, v_1, \dots, v_n\}$  results in an  $(n - 1)$ -simplex, which we denote by  $\tau_i$ . By the definition, every  $(n - 1)$ -simplex can be obtained in this way.

Suppose first that  $i \neq 0, n$ . Then  $\tau_i$  contains both  $v_0$  and  $v_n = v_0 + \mathbb{1}$  and hence is not contained in  $\text{bd } K$ . In a pivot sequence different coordinates are increasing by 1 at different steps and therefore

$$v_i = v_{i-1} + \varepsilon_p \quad \text{and} \quad v_{i+1} = v_i + \varepsilon_q$$

for some  $p \neq q$ . Let  $w = v_{i-1} + \varepsilon_q$ . Then  $v_{i+1} = w + \varepsilon_p$  and hence replacing  $v_i$  by  $w$  in the sequence  $v_0, v_1, \dots, v_n$  results in a pivot sequence satisfying. This, in turn, implies that replacing  $v_i$  by  $w$  in  $\sigma$  results in an  $n$ -simplex having  $\tau_i$  as a face. Since in a pivot sequence different coordinates are increasing at different steps,  $v_i$  can be replaced only by  $w$ . Therefore there are exactly two  $n$ -simplices having  $\tau_i$  as a face. This completes the proof in the case when  $i \neq 0, n$ .

Suppose now that  $i = 0$ . If  $j > 0$ , then  $v_j = v_{j-1} + \varepsilon_p$  for some  $p$ , and hence there are no vertices  $w$  such that  $v_{j-1} < w < v_j$ . It follows that if replacing the vertex  $v_0$  in  $\sigma$  by  $w \neq v_0$  results in an  $n$ -simplex, then either  $w < v_1$ , or  $v_n < w$ . Let  $q$  be such that

$$v_1 = v_0 + \varepsilon_q.$$

Since different coordinates are increasing at different steps, if  $w < v_1$ , then  $v_1 = w + \varepsilon_q$  and hence  $w = v_0$ , contrary to the assumption. By the same reason, if  $v_n < w$ , then  $w = v_n + \varepsilon_q$ . Therefore, if  $v_n + \varepsilon_q \notin K$ , then  $\sigma$  is the only  $n$ -simplex containing  $\tau_0$ . But  $v_n + \varepsilon_q \notin K$  only if the  $q$ th coordinate of  $v_n$  is equal to  $k$ . Since the  $q$ th coordinate does not change along the sequence  $v_1, v_2, \dots, v_n$ , in this case  $\tau_0 \subset \mathcal{B}_q \subset \text{bd } K$ .

Suppose now that  $v_n + \varepsilon_q \in K$ . Then  $\tau_0 \not\subset \mathcal{B}_q$ . Also,  $\tau_0 \not\subset \mathcal{A}_q$  because  $v_1 \in \tau_0$  and  $v_1 = v_0 + \varepsilon_r$ , and  $\tau_0$  cannot be contained in  $\mathcal{A}_p$  or  $\mathcal{B}_p$  for  $p \neq q$  because only the  $q$ th coordinate does not change along the sequence  $v_1, v_2, \dots, v_n$ . At the same time

$$\{v_1, v_2, \dots, v_n, v_n + \varepsilon_q\}$$

is the second  $n$ -simplex having  $\tau_0$  as a face and there are no other such simplices. Therefore in this case  $\tau_0$  is not contained in  $\text{bd } K$  and is a face of exactly two  $n$ -simplices. This completes the proof in the case of  $i = 0$ . The case of  $i = n$  is completely similar. ■

**Kuhn's proof of Kuhn's strong lemma.** Let  $\Delta$  be the abstract simplicial complex having  $1, 2, \dots, n+1$  as its vertices and all subsets of  $\{1, 2, \dots, n+1\}$  as its simplices. The reduced labeling  $r$  can be considered as a simplicial map  $K \rightarrow \Delta$ . Let  $e$  be the number of  $n$ -simplices  $\sigma$  of  $K$  such that  $r(\sigma) = \Delta$ . We need to prove that  $e$  is odd.

Let  $\Delta_1$  be the  $(n-1)$ -face  $\{2, \dots, n+1\}$  of  $\Delta$ , and let  $h$  be the number of  $(n-1)$ -simplices  $\tau$  of  $K$  such that  $\tau \subset \mathcal{B}_1$  and  $r(\tau) = \Delta_1$ . We claim that  $e \equiv h$  modulo 2. Clearly,  $r(v) \leq i$  if  $v \in \mathcal{A}_i$ , and  $r(v) \neq i$  if  $v \in \mathcal{B}_i$ . It follows that  $r(\tau) \neq \Delta_1$  if  $\tau$  is an  $(n-1)$ -simplex of  $K$  contained in  $\text{bd}K$  but not in  $\mathcal{B}_1$ . Using Lemma 9.1, we can follow now either the combinatorial or a cochain-based proof of Sperner's lemma and conclude that  $e \equiv h$  modulo 2.

By forgetting the first coordinate we can identify  $\mathcal{B}_1$  with the cube defined in the same way as  $K$ , but with  $n-1$  in the role of  $n$ . Let us consider the map  $\mathcal{B}_1 \rightarrow \{0, 1\}^{n-1}$  having  $l_2, \dots, l_n$  as its components. Then  $r': v \mapsto r(v) - 1$  is the corresponding reduced labeling. Obviously,  $h$  is equal to the number of  $(n-1)$ -simplices  $\tau$  of  $\mathcal{B}_1$  such that  $r'(\tau) = \{1, 2, \dots, n\}$ . By using an induction by  $n$  we can assume that  $h$  is odd (the case of  $n = 0$  being trivial). By the previous paragraph this implies that  $e$  is odd. ■

**Kuhn's strong lemma, Hurewicz's theorems, and Lebesgue tilings.** Let  $e_i = r^{-1}(i)$ , where  $i = 1, 2, \dots, n+1$ . Clearly, the sets  $e_1, e_2, \dots, e_{n+1}$  are pairwise disjoint and their union is equal to  $K$ . The assumptions of Kuhn's lemma imply that

$$\mathcal{A}_i \subset e_1 \cup \dots \cup e_i$$

and  $e_i$  is disjoint from  $\mathcal{B}_i$  for every  $i \leq n$ . If  $i > j \geq 1$ , then  $e_i$  is disjoint from  $\mathcal{A}_j$  because  $\mathcal{A}_j \subset e_1 \cup \dots \cup e_j$  and the sets  $e_1, e_2, \dots, e_{n+1}$  are pairwise disjoint. Therefore  $e_1, e_2, \dots, e_{n+1}$  satisfy the assumptions of Theorem 4.1 and this theorem implies Kuhn's strong lemma. In other words, Kuhn's strong lemma is an immediate corollary of Hurewicz's theorems as applied to Lebesgue tilings. Apparently, this connection wasn't noticed before.

**Kuhn's lemmas and Lebesgue results.** In order to compare Kuhn's results with Lebesgue ones, one needs to express them in the same language. This is done in Appendix 2, where Lebesgue results are transported from  $Q$  to  $K$ . It seems that the only essential conclusion of Kuhn's lemma and Theorem 7.7 is the existence of an  $n$ -cube intersecting  $c_i$  and  $\bar{c}_i$  for every  $i \in I$ . This part is nothing else but Theorem 1.7 transported from  $Q$  to  $K$ . See Theorem A.2.4. Similarly, it seems that the only essential conclusion of Kuhn's strong lemma is the existence of an  $n$ -cube  $\sigma$  of  $K$  such that  $r(\sigma) = \{1, 2, \dots, n+1\}$ . This is equivalent to the existence of an  $n$ -cube  $\sigma$  intersecting every set  $e_i = r^{-1}(i)$ . By the previous subsection, the sets  $e_1, e_2, \dots, e_{n+1}$  satisfy the assumptions of Theorem A.2.3 and hence this theorem implies the existence of such an  $n$ -cube. On the other hand Theorem A.2.3 is the result of transporting Theorem 1.5 from  $Q$  to  $K$ . We see that the essential part of Kuhn's lemmas was contained already in Lebesgue results.

**In search of the lost cubes: II.** The above proof is Kuhn's own proof up to notations and terminology. With the simplicial complex  $K$  at hand, it is a fairly straightforward adaptation of standard proofs of Sperner's lemma. Given Kuhn's goal to replace simplices by cubes, it is hardly satisfactory, and Kuhn [Ku] posed, among other, the following question.

Is it possible to prove the Cubical Sperner Lemma without resorting to a simplicial decomposition of the  $n$ -cube?

Since Freudenthal's simplicial decomposition of the cube is included already in the statement of Kuhn's lemma in the form of the condition (23), this question hardly can have a positive answer, if understood literally. But one can hide the simplices of this decomposition fairly well. A positive answer was suggested by L.A. Wolsey [W]. In [W] the simplices are hidden in the recursive definition of *completely labeled*  $m$ -cubes. See [W], Definition 1.

Theorem 7.7 also may qualify as a positive answer. In its proof the simplices are hidden in the formula for the multiplication of several 1-cochains. But this formula is just a special case of the general formula (18) for the multiplication of several cochains, which immediately follows from the definition of products of two cubical cochains. Neither (18), nor this definition, adapted from the celebrated work of J.-P. Serre [Se], involve any triangulations.

Instead of hiding simplices one can weaken a little the conclusions of Kuhn's lemmas. Namely, one can replace the sets of terms of pivot sequences by  $n$ -cubes. As we saw, these weakened results were essentially proved by Lebesgue 40 years earlier than Kuhn's asked his question.

**Missed opportunities.** At the end of his paper Kuhn wrote that

... cubical complexes have the crucial advantage of their ease of description in binary notation; for digital computation, therefore, they are a natural object to study. This advantage, however, is offset by the absence of an appropriate homology theory with a boundary operator suited to our purposes.

This was written about 10 years after the cubical homology and cohomology theories appeared in the celebrated Serre's paper [Se]. At the time H.W. Kuhn was an associate professor of mathematics and economics at Princeton and a colleague of J.W. Milnor and J.C. Moore (see [GO]). Serre's theory and other methods of French school was "a Princeton speciality", and J.C. Moore was a leading proponent of these methods (see [BGK]). A.W. Tucker, a pupil of S. Lefschetz, was the chairman of Princeton Mathematics Department from 1952 to 1963.

In 1960 H.W. Kuhn and A.W. Tucker were experts in game theory and mathematical programming. Still, even the title of Kuhn's paper [Ku] indicates that it deals with topology. Did he ever discuss it with a topologist? The situation resembles a story related by F. Dyson in his famous and influential essay [Dy]. Being a physicist, Dyson did not discuss his ideas in number theory with I. MacDonald, a mathematician, who happened to work at the same time and place on almost the same questions.

## 10. Ky Fan's cubical Sperner lemma

**Adjacency-preserving maps.** Let us keep the notations of Section 9. Two points  $v, w$  of  $K$  or  $\mathbb{K}$  are said to be *adjacent* if  $\{v, w\}$  is a 1-cube. A map  $\varphi: K \rightarrow \mathbb{K}$  is said to be *adjacency-preserving* if it takes adjacent vertices of  $K$  to adjacent or equal vertices of  $\mathbb{K}$ . Ky Fan [Fa] called such maps *cubical vertex map*, but in the opinion of the present author these map do not really deserve such name. The notion of adjacency-preserving maps is *not* a genuine cubical analogue of the notion of simplicial maps. For example, the image of a cube of  $K$  under an adjacency-preserving map is not a cube of  $\mathbb{K}$  in general.

**Symmetries.** A *reflection* of the unit discrete cube  $\mathbb{K}$  is a bijection  $\mathbb{K} \rightarrow \mathbb{K}$  acting independently on each of  $n$  coordinates by maps equal either to  $a \mapsto 1 - a$  or to the identity  $a \mapsto a$ . A *symmetry* of  $\mathbb{K}$  is a composition of a reflection and a permutation of coordinates. A *reflection* or a *symmetry* of an  $n$ -cube  $\sigma$  of  $K$  is a bijection  $\sigma \rightarrow \sigma$  such that identifying  $\sigma$  with  $\mathbb{K}$  by a translation turns it into a reflection or a symmetry of  $\mathbb{K}$  respectively.

**Opposite vertices and free pivot sequences.** Let  $\tau$  be an  $m$ -cube of  $K$ . Two vertices of  $\tau$  are said to be *opposite* if they differ in  $m$  coordinates. A *free pivot sequence* of vertices of  $\tau$  is a sequence  $v_0, v_1, \dots, v_m \in \tau$  such that only one coordinate changes when one passes from  $v_{i-1}$  to  $v_i$  and different coordinates change at different steps. Clearly,  $v_0$  and  $v_m$  are opposite vertices of  $\tau$ , and two opposite vertices can be connected by a free pivot sequence.

Suppose now that  $m = n$ . In this case, if  $s(i)$  is the number of the coordinate changing from  $v_{i-1}$  to  $v_i$ , then  $s$  is a permutation of  $1, 2, \dots, n$ , and  $v_0, v_1, \dots, v_n$  is uniquely determined by  $v_0$  and  $s$ . Also, if  $m = n$ , then a sequence  $v_0, v_1, \dots, v_n \in \tau$  is a free pivot sequence if and only if some reflection of  $\tau$  turns it into a pivot sequence.

**Adjacency-preserving maps of cubes.** Let us fix an  $n$ -cube  $\sigma$  of  $K$ . Let  $\lambda^0 \sigma = \{r\}$  and  $\lambda^1 \sigma = \{p\}$  be the root and the peak of  $\sigma$  respectively. Let us fix also an adjacency-preserving map  $\varphi: \sigma \rightarrow \mathbb{K}$ . We are interested when  $\varphi$  is a bijection.

**10.1. Lemma.** *Suppose that  $\varphi(r) = 0$  and  $\varphi(p) = \mathbb{1}$ . Then  $\varphi$  takes every pivot sequence to a pivot sequence.*

**Proof.** After identifying  $\sigma$  with  $\mathbb{K}$  by a translation we may assume that  $\sigma = \mathbb{K}$ . Then  $r = 0$  and  $p = \mathbb{1}$ . Every pivot sequence in  $\mathbb{K}$  starts with  $0$  and ends with  $\mathbb{1}$ . Let

$$0 = v_0, v_1, \dots, v_n = \mathbb{1}$$

be a pivot sequence. Then  $\{v_{i-1}, v_i\}$  is a 1-cube for every  $i \geq 1$ . Since  $\varphi$  is adjacency-preserving, this implies that no more than one coordinate changes when one passes from

$\varphi(v_{i-1})$  to  $\varphi(v_i)$ , and the changing coordinate, if any, changes by 1. Since we have to get from 0 to  $\mathbb{1}$  in  $n$  such steps, some coordinate should increase by 1 at each step and different coordinates should increase at different steps. Therefore  $\varphi(v_0), \varphi(v_1), \dots, \varphi(v_n)$  is a pivot sequence. ■

**10.2. Corollary.** *If  $\varphi$  takes some pivot sequence to a pivot sequence, then  $\varphi$  takes every pivot sequence to a pivot sequence.*

**Proof.** Every pivot sequence in  $\sigma$  starts at  $r$  and ends at  $p$ , and every pivot sequence in  $\mathbb{K}$  starts at 0 and ends at  $\mathbb{1}$ . ■

**10.3. Lemma.** *If  $\varphi$  is bijective, then  $\varphi$  takes some free pivot sequence to a pivot sequence.*

**Proof.** Let  $v = \varphi^{-1}(0)$  and  $w = \varphi^{-1}(\mathbb{1})$ . Suppose that  $w$  differs from  $v$  in  $m$  coordinates. Then there is a sequence

$$v = v_0, v_1, \dots, v_m = w$$

of points in  $\sigma$  such that  $v_i$  differs from  $v_{i-1}$  in only one coordinate for every  $i \leq m$ . Equivalently,  $\{v_{i-1}, v_i\}$  is a 1-cube for every  $i \leq m$ . Since  $\varphi$  is adjacency-preserving, this implies that  $\varphi(v_m)$  differs from  $\varphi(v_0)$  in  $\leq m$  coordinates. But actually  $\varphi(v_m) = \mathbb{1}$  and  $\varphi(v_0) = 0$  differ in  $n$  coordinates. Therefore  $m = n$ , i.e.  $w = v_n$  differs from  $v = v_0$  in  $n$  coordinates. In turn, this implies that different coordinates are changed at different steps and hence  $v_0, v_1, \dots, v_n$  is a free pivot sequence. By the same reason

$$\varphi(v_0), \varphi(v_1), \dots, \varphi(v_n)$$

is a free pivot sequence. Since all  $n$  coordinates are increasing between  $\varphi(v_0) = 0$  and  $\varphi(v_n) = \mathbb{1}$ , it is actually a pivot sequence. ■

**10.4. Lemma.** *If  $\varphi$  is bijective, then  $\varphi$  takes some pivot sequence to a free pivot sequence.*

**Proof.** As above, we may assume that  $\sigma = \mathbb{K}$ . Since  $\varphi$  is bijective and adjacency-preserving,  $\varphi$  induces a bijective self-map of the set of 1-cubes of  $\mathbb{K}$ . It follows that  $\varphi^{-1}$  is a bijective adjacency-preserving map. It remains to apply Lemma 10.3 to  $\varphi^{-1}$ . ■

**10.5. Corollary.** *Let  $v \in \sigma$ . If  $\varphi$  is bijective, then  $\varphi$  induces a one-to-one correspondence between the free pivot sequences starting at  $v$  and the free pivot sequences starting at  $\varphi(v)$ .*

**Proof.** After composing  $\varphi$  with a reflection we may assume that  $v$  is the  $<$ -minimal point of  $\sigma$ . Then the free pivot sequences starting at  $v$  are simply pivot sequences. Lemma 10.4

implies that there is a reflection  $\rho$  such that  $\rho \circ \varphi$  takes some pivot sequence to a pivot sequence. Now Corollary 10.2 implies that  $\rho \circ \varphi$  takes every pivot sequence to a pivot sequence, and hence  $\varphi$  takes every pivot sequence to the image of a pivot sequence under  $\rho^{-1} = \rho$ . Such an image is a free pivot sequence starting at  $\varphi(v)$ . This proves the first statement of the lemma. The second follows from the first and the fact that  $\varphi$  is a bijection. ■

**Cochains associated with  $\varphi$ .** Let  $h_i : \sigma \rightarrow \{0, 1\}$ , where  $i = 1, 2, \dots, n$ , be the components of  $\varphi$  as a map to  $\mathbb{K} = \{0, 1\}^n$ . Let us identify  $\{0, 1\}$  with  $\mathbb{F}_2$  and consider the maps  $h_i$  also as 0-cochains of  $\sigma$ . Let

$$f_i = \partial^* h_i.$$

If  $\tau = \{v, w\} \subset \sigma$  is a 1-cube, then  $f_i(\tau) = 1$  if and only if  $h_i(v) \neq h_i(w)$ .

Two points of  $\mathbb{K}$  are adjacent if and only if they differ in no more than one coordinate. Therefore, the adjacency-preserving property of  $\varphi$  means that for every 1-cube  $\tau \subset \sigma$  there is no more than one  $i$  such that  $f_i(\tau) = 1$ . Equivalently, all components  $h_i$  of  $\varphi$ , except, perhaps, one of them, are constant on  $\tau$ .

**10.6. Theorem.** *If  $\varphi$  is bijective, then  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$ .*

**Proof.** Let  $v_0, v_1, \dots, v_n \in \sigma$  be a pivot sequence. It defines a sequence  $\tau_1, \tau_2, \dots, \tau_n$  of 1-cubes, where  $\tau_i = \{v_{i-1}, v_i\}$  for each  $i \geq 1$ . Suppose that

$$(24) \quad f_1(\tau_1) \cdot f_2(\tau_2) \cdot \dots \cdot f_n(\tau_n) = 1.$$

Then  $f_i(\tau_i) = 1$  for every  $i$ . This means that  $\varphi(v_i)$  differs from  $\varphi(v_{i-1})$  in  $i$ th coordinate. Since  $\varphi$  is adjacency-preserving,  $\{\varphi(v_{i-1}), \varphi(v_i)\}$  is a 1-cube and hence  $\varphi(v_i)$  differs from  $\varphi(v_{i-1})$  *only* in  $i$ th coordinate. It follows that the sequence

$$\varphi(v_0), \varphi(v_1), \dots, \varphi(v_n)$$

is an free pivot sequence and the corresponding permutation  $s$  is the identity permutation. Therefore this sequence is uniquely determined by  $\varphi(v_0)$  and the property (24). In view of Corollary 10.5, this implies that the pivot sequence  $v_0, v_1, \dots, v_n \in \sigma$  is uniquely determined by the property (24), i.e. there is exactly one pivot sequence  $v_0, v_1, \dots, v_n \in \sigma$  such that (24) holds.

As we saw in Section 7, the value  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma)$  is equal to the sum of all products  $f_1(\tau_1) \cdot f_2(\tau_2) \cdot \dots \cdot f_n(\tau_n)$  corresponding to pivot sequences. Each such product is equal either to 1 or to 0. By the previous paragraph, this product is equal to 1 for exactly one pivot sequence. It follows that  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$ . ■

**10.7. Lemma.** *Let  $\tau$  be an  $(n-1)$ -face of  $\sigma$  such that*

$$(25) \quad f_2 \cdot f_3 \cdot \dots \cdot f_n(\tau) = 1.$$

*Then  $\varphi$  maps  $\tau$  into  $\mathbb{A}_1 = 0 \times \{0, 1\}^{n-1}$  or  $\mathbb{B}_1 = 0 \times \{0, 1\}^{n-1}$ . Moreover,*

$$\varphi(\tau) \subset \mathbb{B}_1$$

*if and only if  $h_1(\lambda^0 \tau) = 1$ , where  $\lambda^0 \tau$  is the root of  $\tau$ .*

**Proof.** Let  $\lambda^0 \tau = \{v\}$  and  $\lambda^1 \tau = \{w\}$  be the root and the peak of  $\tau$ . The equality (25) implies that there is a free pivot sequence  $v = v_1, v_2, \dots, v_n = w$  in  $\tau$  such that

$$f_2(\tau_2) \cdot f_3(\tau_3) \cdot \dots \cdot f_n(\tau_n) = 1,$$

where  $\tau_i = \{v_{i-1}, v_i\}$  for each  $i \geq 2$ . It follows that  $f_i(\tau_i) = 1$  and hence  $\varphi(v_i)$  differs from  $\varphi(v_{i-1})$  in the  $i$ th coordinate for every  $i$ . Since  $\varphi$  is adjacency-preserving,  $\varphi(v_i)$  cannot differ from  $\varphi(v_{i-1})$  in any other coordinate, in particular, in the first one. It follows that the first coordinate of  $\varphi(v_i)$  is independent of  $i$  and that  $\varphi(w) = \varphi(v_n)$  differs from  $\varphi(v) = \varphi(v_0)$  in all other coordinates. In other words,  $\varphi(v)$  and  $\varphi(w)$  both belong either to  $\mathbb{A}_1$  or  $\mathbb{B}_1$  and are opposite vertices of  $\mathbb{A}_1$  or  $\mathbb{B}_1$  respectively.

Suppose now that  $v = w_1, w_2, \dots, w_n = w$  is some other free pivot sequence in  $\tau$  with the same starting and ending points. Since  $\varphi$  is adjacency-preserving,  $\varphi(w_i)$  cannot differ from  $\varphi(w_{i-1})$  in more than one coordinate. Since  $\varphi(w_n) = \varphi(w)$  differs from  $\varphi(w_0) = \varphi(v)$  in all coordinates except the first one, it follows that the first coordinate of  $\varphi(w_i)$  is independent of  $i$  and hence is equal to the first coordinate of  $\varphi(v)$ . Since every point of  $\tau$  is a term of a pivot sequence starting at  $v$  and ending at  $w$ , every point of  $\varphi(\tau)$  has the same first coordinate as  $\varphi(v)$ . Hence  $\varphi(\tau)$  is contained in either  $\mathbb{A}_1$  or  $\mathbb{B}_1$ .

This proves the first statement of the lemma. The second statement follows from the first one and the fact that the root  $\lambda^0 \tau$  is a 0-face of  $\tau$ . ■

**10.8. Theorem.** *If  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$ , then  $\varphi$  is a bijection.*

**Proof.** After identifying  $\sigma$  with  $\mathbb{K}$  by a translation we may apply Theorem 7.4 to  $\sigma$  in the role of  $K$  and the cochains  $h_1, h_2, \dots, h_n$ . Clearly, in this situation  $s = 1$ . Also, every  $(n-1)$ -cube contained in  $\sigma$  is automatically contained in  $\text{bd } \sigma$  and is an  $(n-1)$ -face of  $\sigma$ .

Therefore Theorem 7.4 implies that there is an odd number of  $(n-1)$ -faces  $\tau$  of  $\sigma$  such that  $h_1(\lambda^0 \tau) = 1$  and (25) holds. In view of Lemma 10.7 this means that the number of  $(n-1)$ -faces  $\tau$  of  $\sigma$  such that  $\varphi(\tau) \subset \mathbb{B}_1$  and (25) holds is odd.



We claim that the number  $N$  of  $(n-1)$ -faces  $\tau$  such that (25) holds is even. By Lemma 7.2

$$\partial^*(h_2 \cdot f_3 \cdot \dots \cdot f_n) = f_2 \cdot f_3 \cdot \dots \cdot f_n.$$

The identity  $\partial^* \circ \partial^* = 0$  implies that  $\partial^*(f_2 \cdot f_3 \cdot \dots \cdot f_n) = 0$  and hence

$$\begin{aligned} 0 &= \partial^*(f_2 \cdot f_3 \cdot \dots \cdot f_n)(\sigma) \\ &= f_2 \cdot f_3 \cdot \dots \cdot f_n(\partial\sigma) = \sum_{\tau} f_2 \cdot f_3 \cdot \dots \cdot f_n(\tau), \end{aligned}$$

where the sum is taken over all  $(n-1)$ -faces  $\tau$  of  $\sigma$ . It follows that  $N$  is indeed even. So, the number of  $(n-1)$ -faces  $\tau$  such that (25) holds is even, and there is an odd number of faces  $\tau$  such that  $\varphi(\tau) \subset \mathbb{B}_1$  among them. Therefore, there is also an odd number of faces  $\tau$  such that  $\varphi(\tau) \subset \mathbb{A}_1$  among them. In particular, there exist  $(n-1)$ -faces  $\tau_{\mathbb{A}}$  and  $\tau_{\mathbb{B}}$  such that  $\varphi(\tau_{\mathbb{A}}) \subset \mathbb{A}_1$  and  $\varphi(\tau_{\mathbb{B}}) \subset \mathbb{B}_1$ . Clearly,  $\tau_{\mathbb{A}}$  and  $\tau_{\mathbb{B}}$  are disjoint.

Now it is time to apply an induction by  $n$ . The theorem is trivially true for  $n = 1$ . Arguing by induction, we may assume that it is true with  $n-1$  in the role of  $n$ . Let  $\tau$  be an  $(n-1)$ -face of  $\sigma$  such that (25) holds. Consider the map  $\psi: \tau \rightarrow \{0, 1\}^{n-1}$  having  $h_2, h_3, \dots, h_n$  as its components. The inductive assumption implies that  $\psi$  is a bijection. Together with Lemma 10.7 this implies that  $\varphi$  maps  $\tau$  bijectively onto either  $\mathbb{A}_1$  or  $\mathbb{B}_1$ . By applying this result to  $\tau = \tau_{\mathbb{A}}$  and  $\tau = \tau_{\mathbb{B}}$  we see that  $\varphi$  maps  $\tau_{\mathbb{A}}$  and  $\tau_{\mathbb{B}}$  bijectively onto  $\mathbb{A}_1$  and  $\mathbb{B}_1$  respectively. Since  $\tau_{\mathbb{A}}$  and  $\tau_{\mathbb{B}}$  are two disjoint  $(n-1)$ -face of  $\sigma$ , it follows that  $\varphi$  is a bijection. ■

**10.9. Theorem.** *The map  $\varphi$  is a bijection if and only if  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$ . ■*

**Ky Fan's lemma.** *Let  $\Phi: K \rightarrow \mathbb{K}$  be an adjacency-preserving map, and let  $\mathbb{B}$  be some  $(n-1)$ -face of  $\mathbb{K}$ . Let  $s$  be the number of  $n$ -cubes of  $K$  which are mapped by  $\Phi$  bijectively onto  $\mathbb{K}$ , and let  $t$  be the number of  $(n-1)$ -cubes of  $K$  contained in  $\text{bd}K$  and bijectively mapped by  $\Phi$  onto  $\mathbb{B}$ . Then  $s \equiv t$  modulo 2.*

**Proof.** We may assume that  $\mathbb{B} = \mathbb{B}_1 = 1 \times \{0, 1\}^{n-1}$ . Let  $h_1, h_2, \dots, h_n$  be the components of  $\Phi$  considered as 0-cochains of  $K$ , and let  $f_i = \partial^* h_i$ . Theorem 10.9 implies that  $s$  is equal to the number of  $n$ -cubes  $\sigma$  of  $K$  such that  $f_1 \cdot f_2 \cdot \dots \cdot f_n(\sigma) = 1$ . By combining Theorem 10.9 with  $n-1$  in the role of  $n$  with Lemma 10.7 we see that  $t$  is equal to the number of  $(n-1)$ -cubes  $\tau \subset \text{bd}K$  such that  $h_1(\lambda^0 \tau) = 1$  and (25) holds. These observations show that the theorem is a special case of Theorem 7.4. ■

**Corollary.** *Let  $L: K \rightarrow \mathbb{K}$  be an adjacency-preserving map satisfying the assumptions of Kuhn's lemma. Then the number of  $n$ -cubes of  $K$  mapped by  $L$  bijectively onto  $\mathbb{K}$  is odd.*

**Proof.** It is sufficient to use Ky Fan theorem together with an induction by  $n$ . ■

**Adjacency-preserving and transversality.** Let  $\Phi: K \rightarrow \mathbb{K}$  be an arbitrary map and let  $h_1, h_2, \dots, h_n$  be its components. As in Section 9, let  $c_i$  be the set of points  $v \in K$  such that  $h_i(v) = 0$ . Then  $\Phi$  is determined by the sets  $c_1, c_2, \dots, c_n$ . Let  $\bar{c}_i = K \setminus c_i$ . Considering the maps  $h_i$  as 0-chains of  $K$  allows us define 1-cochains  $f_i = \partial^* h_i$ .

Let  $Q$  be the solid cube of size  $l = k + 1$ , where  $k$  is the size of  $K$ . Let

$$s_i = *c_i \quad \text{and} \quad \bar{s}_i = *\bar{c}_i$$

be the cubic subsets of  $Q$  dual in the sense of Appendix 2 to  $c_i$  and  $\bar{c}_i$  respectively. For every  $i = 1, 2, \dots, n$  let us consider the formal sum  $\gamma_i$  of  $(n-1)$ -cubes  $\tau$  of  $Q$  such that  $\tau$  is a common face of an  $n$ -cube contained in  $s_i$  and an  $n$ -cube contained in  $\bar{s}_i$ . In a natural sense  $\gamma_i$  is the common boundary of the cubical sets  $s_i$  and  $\bar{s}_i$ . One may also say that  $\gamma_i$  is the internal part of the boundary of  $s_i$ .

If  $\tau = \{v, w\}$  is a 1-cube, then  $f_i(\tau) = 1$  if and only if one of the  $n$ -cubes  $*v, *w$  is contained in  $*c_i$  and the other does not. By Lemma 2 the dual  $(n-1)$ -cube  $*\tau$  is the common face of  $*v$  and  $*w$ . It follows that  $*\gamma_i = f_i$ .

As we already saw,  $\Phi$  is adjacency-preserving if and only if for every 1-cube  $\tau$  there is no more than one  $i$  such that  $f_i(\tau) = 1$ . In the language of  $Q$  this means that every  $(n-1)$ -cube of  $Q$  enters no more than one  $\gamma_i$ . In other words, if  $i \neq j$ , then  $\gamma_i$  and  $\gamma_j$  have no common  $(n-1)$ -cubes. If one thinks about  $\gamma_i$  and  $\gamma_j$  as hypersurfaces, then this means that they are nowhere tangent. Two smooth hypersurfaces are nowhere tangent if and only if they are transverse. We see that the adjacency-preserving property is a sort of transversality.

The pivot sequences  $v_0, v_1, \dots, v_n$  such that  $\{v_0, v_1, \dots, v_n\}$  intersects both  $c_i$  and  $\bar{c}_i$  for every  $i$  correspond to the points of intersection of all hypersurfaces  $\gamma_i$ , and Kuhn's lemma ensures that these hypersurfaces do intersect. But it provides no information about *how* they intersect. As is well known, under appropriate transversality assumptions the intersections can be understood better than otherwise. If  $\Phi$  is adjacency-preserving, then Ky Fan theorem ensures the existence of cubes  $\sigma$  of  $K$  such that  $\Phi$  maps  $\sigma$  bijectively onto  $\mathbb{K}$ . Such cubes also correspond to the points of intersection of all hypersurfaces  $\gamma_i$  (in fact, every pivot sequence from Kuhn's lemma is contained in such a cube). The bijectivity tells us that  $2^n$  cubes  $c$  meeting at such point of intersection realize all  $2^n$  compatible combinations of conditions  $c \subset s_i, c \subset \bar{s}_j$  (where  $1 \leq i, j \leq n$ ). Therefore at least in this respect the intersection of the hypersurfaces defined by an adjacency-preserving map  $\Phi$  is similar to the simplest case, the intersection of  $n$  hyperplanes parallel to the coordinate hyperplanes.

**A question of Kuhn.** Kuhn [Ku] asked if there is any relation between Kuhn's lemma and Ky Fan's theorem. The author believes that the above discussion provides an answer.

## A.1. Freudenthal's triangulations of cubes and simplices

**The unit cube**  $[0, 1]^n$ . As usual, let  $I = \{1, 2, \dots, n\}$ . For every permutation (i.e bijection)  $\omega: I \longrightarrow I$  let  $\Delta_\omega$  be the set of points  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  such that

$$(26) \quad 1 \geq x_{\omega(1)} \geq x_{\omega(2)} \geq \dots \geq x_{\omega(n)} \geq 0.$$

Obviously, the cube  $[0, 1]^n$  is equal to the union of the sets  $\Delta_\omega$  over all permutations  $\omega$ . If  $\omega$  is the identity, then  $\Delta = \Delta_\omega$  is the set of points  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  such that

$$(27) \quad 1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0.$$

**A.1.1. Lemma.**  $\Delta$  is the geometric  $n$ -simplex with the vertices  $u_0, u_1, \dots, u_n \in \mathbb{R}^n$ , where

$$u_i = (1, \dots, 1, 0, \dots, 0)$$

is the point with the first  $i$  coordinates equal to 1 and the last  $n - i$  coordinates equal to 0. Every face of  $\Delta$  is defined by (27) with several inequality signs  $\geq$  replaced by  $=$ .

**Proof.** Indeed, every  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  has a unique presentation in the form

$$(28) \quad (x_1, x_2, \dots, x_n) = y_0 u_0 + y_1 u_1 + \dots + y_{n-1} u_{n-1} + y_n u_n,$$

where  $y_0, y_1, \dots, y_n \in \mathbb{R}$  and

$$y_0 + y_1 + \dots + y_n = 1.$$

In fact, (28) holds if and only if

$$x_i = y_i + y_{i+1} + \dots + y_n$$

for all  $i = 1, 2, \dots, n$  and hence if and only if

$$y_n = x_n \quad \text{and} \quad y_i = x_i - x_{i+1}$$

for all  $i \leq n - 1$ . These equalities imply that  $x_1 = y_1 + y_2 + \dots + y_n$  and  $y_0 = 1 - x_1$ . It follows that  $y_0, y_1, \dots, y_n \geq 0$  if and only if (27) holds and hence  $\Delta$  is the convex hull of the points  $u_0, u_1, \dots, u_n$ . Clearly, these points are affinely independent and hence  $\Delta$  is indeed a geometric simplex and has these points as its vertices. Every face of  $\Delta$  is defined by requiring several barycentric coordinates  $y_i$  to be 0. But the equality  $y_i = 0$  is equivalent to  $x_n = 0$  if  $i = n$ , to  $x_1 = 1$  if  $i = 0$ , and to  $x_i = x_{i+1}$  if  $0 < i < n$ . ■

**A.1.2. Corollary.**  $\Delta_\omega$  is a geometric  $n$ -simplex for every  $\omega$ . Its vertices  $v_0, v_1, \dots, v_n$  belong to  $\{0, 1\}^n$  and can be ordered in such a way that

$$0 = v_0 < v_1 < \dots < v_n = \mathbb{1},$$

where  $<$  is the order from Sections 2 and 9. Every face of  $\Delta_\omega$  is defined by (26) with several inequality signs  $\geq$  replaced by  $=$ .

**Proof.** Obviously, the vertices  $u_0, u_1, \dots, u_n$  of  $\Delta$  belong to  $\{0, 1\}^n$ . Moreover,

$$0 = u_0 < u_1 < \dots < u_n = \mathbb{1}.$$

Every set  $\Delta_\omega$  is the image of  $\Delta$  under a map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  permuting coordinates. It remains to notice that such maps preserve the order  $<$  and other relevant properties. ■

**A.1.3. Corollary.** For every  $\omega, \omega'$  the intersection  $\Delta_\omega \cap \Delta_{\omega'}$  is a common face of the simplices  $\Delta_\omega$  and  $\Delta_{\omega'}$ .

**Proof.** Clearly,  $\Delta_\omega \cap \Delta_{\omega'}$  can be defined by (26) with several signs  $\geq$  replaced by  $=$ . ■

**The canonical triangulations of unit cubes.** Since  $[0, 1]^n$  is equal to the union of simplices  $\Delta_\omega$ , Corollary A.1.3 implies that simplices  $\Delta_\omega$  together with their faces form a triangulation of the unit cube  $[0, 1]^n$ . This is the *canonical triangulation* of  $[0, 1]^n$ . It is invariant under all permutations of the coordinates, but not under most of other symmetries of  $[0, 1]^n$ . An  $m$ -cube  $c$  of  $Q$  can be identified with  $[0, 1]^m$  by a translation followed by a permutation of coordinates. Clearly, the involved translation is uniquely determined by  $c$ . One can use such an identification to transplant the canonical triangulation of  $[0, 1]^m$  to  $c$ . Since the canonical triangulation of  $[0, 1]^m$  is invariant under permutation of coordinates, the result does not depend on the choice of identification. It is called the *canonical triangulation* of  $c$ .

**A.1.4. Lemma.** If  $c$  is a face of  $[0, 1]^n$ , then the collection of simplices of the canonical triangulation of  $[0, 1]^n$  contained in  $c$  is the canonical triangulation of  $c$ .

**Proof.** The face  $c$  is defined by requiring some coordinates to be 0 and some other to be 1. After permuting coordinates we may assume that the coordinates required to be 0 are the last ones, and required to be 1 are the first ones. In this case the lemma is trivial. ■

**The canonical triangulation of  $Q$ .** We see that the canonical triangulations of cubes agree on common faces. Hence the union of these canonical triangulations is a triangulation of  $Q$ . It is called the *canonical triangulation* of  $Q$ . The corresponding abstract simplicial complex has as its set of vertices the discrete cube of the same size as  $Q$ . Corollary A.1.2 implies that it is nothing else but the abstract simplicial complex from Section 9.

**Freudenthal's triangulations of standard simplices.** We will assume that the size  $l$  of the big cube  $Q$  is  $\geq 2$ . Let  $l\Delta$  the simplex defined by the inequalities (27) with 1 replaced by  $l$ . Alternatively,  $l\Delta$  is the image of  $\Delta$  under the map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  multiplying each coordinate by  $l$ . As we will see in a moment (see Theorem A.1.5 and Corollary A.1.6 below),  $l\Delta$  is equal to the union of simplices of the canonical triangulation of  $Q$  contained in  $l\Delta$ . Therefore these simplices form a triangulation of  $l\Delta$ , which we will call *Freudenthal's triangulation* of  $l\Delta$ . Freudenthal himself [Fr] considered only the case  $l = 2$ .

It will be convenient to use instead of permutations  $\omega$  their inverses. For every permutation  $\omega: I \rightarrow I$  let  $\Delta(\omega) = \Delta_{\omega^{-1}}$ . Then  $\Delta(\omega)$  consists of points  $(x_1, x_2, \dots, x_n) \in [0, 1]^n$  such that  $x_i \geq x_j$  if and only if  $\omega(i) \geq \omega(j)$ . Let  $l\Delta(\omega)$  be the subset of  $Q = [0, l]^n$  defined by the same inequalities. Every simplex of the canonical triangulation of  $Q$  has the form  $a + \Delta(\omega)$ , where  $a = (a_1, a_2, \dots, a_n)$  and  $a_1, a_2, \dots, a_n$  are integers between 0 and  $l - 1$ , and  $\omega$  is a permutation. Let us look when  $a + \Delta(\omega)$  is contained in  $l\Delta(\omega')$ .

**A.1.5. Theorem.** *For every permutation  $\omega'$  the simplex  $a + \Delta(\omega)$  is contained in  $l\Delta(\omega')$  if and only if  $\omega' = \omega|a$ , where  $\omega|a$  is the unique permutation such that*

$$\omega|a(i) > \omega|a(j)$$

*if either  $a_i > a_j$ , or  $a_i = a_j$  and  $\omega(i) > \omega(j)$ . Every simplex  $l\Delta(\omega')$  is equal to the union of simplices of the form  $a + \Delta(\omega)$  contained in it.*

**Proof.** Every point of  $Q$  has the form

$$a + x = (a_1 + x_1, a_2 + x_2, \dots, a_n + x_n),$$

where  $a = (a_1, a_2, \dots, a_n)$  is as above and  $x = (x_1, x_2, \dots, x_n) \in [0, 1]^n$ . Clearly, if  $a_i > a_j$ , then  $a_i + x_i \geq a_j + x_j$ . If  $a_i = a_j$ , then  $a_i + x_i \geq a_j + x_j$  is trivially equivalent to  $x_i \geq x_j$ . It follows that  $a + \Delta(\omega) \subset l\Delta(\omega|a)$ . Since different simplices  $l\Delta(\omega')$  have disjoint interiors,  $a + \Delta(\omega)$  is not contained in any other  $l\Delta(\omega')$ . Moreover, if  $\omega' \neq \omega|a$ , then the intersection of  $a + \Delta(\omega)$  and  $l\Delta(\omega')$  is contained in the boundary of  $l\Delta(\omega')$ . Since  $Q$  is equal to the union of simplices of the form  $a + \Delta(\omega)$ , it follows that  $l\Delta(\omega')$  is equal to the union of simplices  $a + \Delta(\omega)$  contained in  $l\Delta(\omega')$ . ■

**A.1.6. Corollary.** *For every permutation  $\omega'$  the simplices  $a + \Delta(\omega)$  corresponding to pairs  $a, \omega$  such that  $\omega|a = \omega'$ , together with their faces, form a triangulation of  $l\Delta(\omega')$ . ■*

**Remark.** The simplices  $a + \Delta(\omega)$  corresponding to pairs  $a, \omega$  such that  $\omega|a = \text{id}$  triangulate  $l\Delta$ . If  $\omega|a = \text{id}$ , then  $i > j$  implies that  $a_i \geq a_j$ , i.e.  $a_1 \geq a_2 \geq \dots \geq a_n$  and hence  $a \in l\Delta$ . Of course, geometrically this is obvious. Indeed,  $0 \in \Delta(\omega)$  and hence  $a + \Delta(\omega) \subset l\Delta$  implies  $a \in l\Delta$ . Since  $a_i \leq l - 1$ , even more is true:  $a + \mathbb{1} \in l\Delta$ .

**The case  $l = 2$ .** This is the only case considered by Freudenthal [Fr]. If  $l = 2$ , then  $1 \geq a_1 \geq a_2 \geq \dots \geq a_n \geq 0$  and hence there is a number  $q \leq n$  such that  $a_i = 1$  for  $i \leq q$  and  $a_j = 0$  for  $i > q$ . For such  $a$  the condition  $\omega|a = \text{id}$  is equivalent to the following: if either  $i, j \leq q$  or  $i, j > q$ , then  $i > j$  implies  $\omega(i) > \omega(j)$ . Permutations  $\omega$  satisfying this condition are known as  $(p, q)$ -shuffles, where  $p = n - q$ . Clearly, a  $(p, q)$ -shuffle  $\omega$  is determined by the set  $\{\omega(i) \mid i \leq q\}$ . Conversely, this set is trivially determined by  $\omega$  and  $q$ , but not by  $\omega$  alone, as the example of the identity permutation shows. It follows that the simplices of Freudenthal's triangulation of  $2\Delta$  correspond to subsets of  $\{1, 2, \dots, n\}$ . In particular, the number of  $n$ -simplices of this triangulation is  $2^n$ .

The vertices of every  $n$ -simplex of Freudenthal's triangulation of  $2\Delta$  can be obtained as follows. Let the 0th vertex be  $(1, \dots, 1, 0, \dots, 0)$  with  $q$  coordinates 1 followed by  $n - q$  coordinates 0. The 1st vertex results from adding 1 either to the first 1 or to the first 0. Continuing in this way, one adds 1 at each step either to the leftmost 1 among the first  $q$  coordinates, or to the leftmost 0 among the last  $n - q$  coordinates. After  $n$  steps one gets all vertices of a simplex of Freudenthal's triangulation of  $2\Delta$ .

**Freudenthal's triangulations of arbitrary simplices and polyhedra.** Every  $n$ -simplex  $\Gamma$  in a euclidean space  $\mathbb{R}^m$  is the image of  $l\Delta$  under an *affine map*  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ , i.e. a map of the form  $x \mapsto L(x) + a$ , where  $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear map and  $a \in \mathbb{R}^m$ . The image of Freudenthal's triangulation of  $l\Delta$  under such map is a triangulation of  $\Gamma$ . If the vertices of  $\Gamma$  are ordered, then there is a preferred map  $l\Delta \rightarrow \Gamma$ , namely, the map taking the vertex  $lu_i$  of  $l\Delta$  to the  $i$ th vertex of  $\Gamma$ . This leads to a preferred triangulation of  $\Gamma$ .

Let  $S$  is a geometric simplicial complex and let us choose a linear order on the set of its vertices. This order induces an order on the set of vertices of each simplex of  $S$ . The corresponding preferred triangulations of simplices agree on common faces and hence define a triangulation of the polyhedron  $\|S\|$  of  $S$ . For large  $l$  one gets triangulations of  $\|S\|$  into arbitrarily small simplices. Instead of using large  $l$  Freudenthal iterated the construction with  $l = 2$ .

**The geometry of Freudenthal's triangulations.** Every  $n$ -simplex of Freudenthal's triangulation of  $l\Delta$  is the image of  $\Delta$  under a permutation of coordinates followed by a translation. In particular, all  $n$ -simplices of this triangulation are isometric to  $\Delta$  and  $\Delta$  is homothetic to  $l\Delta$ . Less formally, one can say that these simplices have the same shape as  $l\Delta$  and are  $l$  times smaller than  $l\Delta$ . Taking the images under affine maps preserves these properties, and hence Freudenthal's construction leads to triangulations of any polyhedron  $\|S\|$  into into arbitrarily small simplices, each having the same shape as one of the simplices of  $S$ .

Freudenthal's construction [Fr] answered a question of L.E.J. Brouwer about a *simple* construction of triangulations of polyhedra into arbitrarily small simplices which do not eventually include nearly flat simplices, i.e.  $n$ -simplices with arbitrarily small ratio  $V/d^n$ , where  $V$  is the volume and  $d$  is the diameter. In contrast with Freudenthal's triangulations, the iterated barycentric subdivisions eventually include nearly flat  $n$ -simplices.

## A.2. The solid cube and the discrete cube

**A naive approach to relations between  $Q$  and  $K$ .** Let  $Q, l$  and  $K, k$  be as in Sections 1 and 7 respectively. Naively, one would think that one should set  $k = l$  and treat  $K$  as the set of vertices of cubes of  $Q$ . An  $m$ -cube of  $Q$  is determined by its set of vertices, which is an  $m$ -cube of  $K$ , and each  $m$ -cube of  $K$  is the set of vertices of an  $m$ -cube of  $Q$ . This gives as a canonical one-to-one correspondence between  $m$ -cubes of  $Q$  and of  $K$ . In turn, this leads to a canonical one-to-one correspondence between  $m$ -chains of  $Q$  and of  $K$ . A cubical subset of  $Q$  can be identified with the formal sum of  $n$ -cubes contained in it, and hence with an  $n$ -chain. This allows to interpret the cubical subsets of  $Q$  in terms of  $K$ .

The proofs of all results of Section 1 up to Theorem 1.6 (i.e up turning to closed sets) can be straightforwardly translated into purely combinatorial language of  $K$  with exception of the intersections-based proof of Lemma 1.4. One can save the day with the projections-based proof, which admits a straightforward translation. In Lusternik–Schnirelmann context there is no obvious analogue of projections. In fact, the intersections-based proofs also can be translated into the combinatorial language of  $K$ , but at the cost of obscuring the ideas.

**Duality.** The proper way to understand the relations between the proofs using the solid cube  $Q$  and the proofs using the discrete cube  $K$  is different. The discrete counterpart to  $Q$  is not the set of the vertices of  $Q$ , but the set of the centers of  $n$ -cubes of  $Q$ , and the discrete cube  $K$  should be thought of as the set of these centers, for convenience shifted in such a way that the coordinates of the points of  $K$  are integers. Let us set  $k = l - 1$  and consider the translation  $t: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by the vector  $(1/2, 1/2, \dots, 1/2)$ , i.e. the map

$$t: (x_1, x_2, \dots, x_n) \mapsto (x_1 + 1/2, x_2 + 1/2, \dots, x_n + 1/2).$$

Then  $t(K)$  is the set of centers of  $n$ -cubes of  $Q$ . For  $v \in K$  let  $*v$  be the  $n$ -cube of  $Q$  with the center  $t(v)$ . In more details, if  $v = (a_1, a_2, \dots, a_n)$ , then

$$*v = \prod_{i=1}^n [a_i, a_i + 1].$$

The map  $v \mapsto *v$  is a bijection between  $K$  and the set of  $n$ -cubes of  $Q$ . It should be considered as a sort of duality: it assigns to a 0-cube of  $K$  an  $n$ -cube of  $Q$ . Experts will immediately recognize here an instance of Poincaré duality. For  $s \subset K$  let

$$*s = \bigcup_{v \in s} *v.$$

Then  $s \mapsto *s$  is a bijection between subsets of  $K$  and cubical subsets of  $Q$ . The maps  $\bullet \mapsto *\bullet$  allow to translate results about  $K$  into results about  $Q$  and vice versa. Let us begin with simplest examples.

**A.2.1. Lemma.** *The sets  $*\mathcal{A}_i$  and  $*\mathcal{B}_i$  are the unions of all  $n$ -cubes of  $Q$  intersecting  $A_i$  and  $B_i$  respectively. A subset  $s \subset K$  contains  $\mathcal{A}_i$  or  $\mathcal{B}_i$  if and only if the subset  $*s \subset Q$  contains  $A_i$  or  $B_i$  respectively. Similarly, a subset  $s \subset K$  intersects  $\mathcal{A}_i$  or  $\mathcal{B}_i$  if and only if the subset  $*s \subset Q$  intersects  $A_i$  or  $B_i$  respectively.*

**Proof.** The first statement is obvious, and the rest immediately follows from it. ■

**A.2.2. Lemma.** *Let  $s_1, s_2, \dots, s_r$  be subsets of  $K$ . There exists an  $n$ -cube of  $K$  intersecting every set  $s_1, s_2, \dots, s_r$  if and only if  $*s_1 \cap *s_2 \cap \dots \cap *s_r \neq \emptyset$ .*

**Proof.** Let us consider an  $n$ -cube

$$\sigma = \prod_{i=1}^n \{a_i, a_i + 1\}$$

of  $K$ . If  $v \in \sigma$ , then, obviously,  $(a_1 + 1, a_2 + 1, \dots, a_n + 1) \in v^*$ . Therefore

$$*v_1 \cap *v_2 \cap \dots \cap *v_r \neq \emptyset$$

if all  $v_1, v_2, \dots, v_r$  are contained in an  $n$ -cube of  $K$ . It follows that if  $s_1, s_2, \dots, s_r$  intersect  $\sigma$ , then  $*s_1 \cap *s_2 \cap \dots \cap *s_r \neq \emptyset$ . This proves the “only if” part of lemma.

Suppose now that  $*s_1 \cap *s_2 \cap \dots \cap *s_r \neq \emptyset$ . By the definition of the sets  $*s_i$  in this case  $*v_1 \cap *v_2 \cap \dots \cap *v_r \neq \emptyset$  for some points  $v_i \in s_i$ . Suppose that

$$(a_1, a_2, \dots, a_n) \in *v_1 \cap *v_2 \cap \dots \cap *v_r$$

and consider the  $n$ -cube

$$\sigma = \prod_{i=1}^n \{a_i - 1, a_i\}$$

of  $K$ . Clearly,  $v_i \in \sigma$  for every  $i \in I$ . This proves the “if” part of lemma. ■

**A.2.3. Theorem.** *Let  $s_1, s_2, \dots, s_{n+1} \subset K$ . Suppose that  $s_1 \cup s_2 \cup \dots \cup s_{n+1} = K$  and*

- (i)  $s_1 \cup s_2 \cup \dots \cup s_i$  contains  $\mathcal{A}_i$  and  $s_i$  is disjoint from  $\mathcal{B}_i$  for  $i \leq n$ , and
- (ii)  $s_i$  is disjoint from  $\mathcal{A}_j$  if  $i > j$ .

*Then there exists an  $n$ -cube of  $K$  intersecting every set  $s_1, s_2, \dots, s_{n+1}$ .*

**Proof.** This theorem is a duality-based translation of Theorem 1.5 into the language of  $K$ . Let  $e_i = *s_i$  for every  $i = 1, 2, \dots, n + 1$ . Since  $s_1 \cup s_2 \cup \dots \cup s_{n+1} = K$ , the sets



$e_1, e_2, \dots, e_{n+1}$  form a covering of  $Q$ . The assumption (i) together with Lemma A.2.1 imply that  $e_1 \cup e_2 \cup \dots \cup e_i$  contains  $A_i$  and is disjoint from  $B_i$  for  $i \leq n$ . Similarly, the assumption (ii) together with Lemma A.2.1 imply that  $e_i$  is disjoint from  $A_j$  for  $i > j$ . Therefore the sets  $e_1, e_2, \dots, e_{n+1}$  satisfy the assumptions of Theorem 1.5. By this theorem  $e_1 \cap e_2 \cap \dots \cap e_{n+1} \neq \emptyset$ . It remains to apply Lemma A.2.2. ■

**A.2.4. Theorem.** *Let  $s_1, s_2, \dots, s_n \subset K$  and  $\bar{s}_i = K \setminus s_i$ . If  $\mathcal{A}_i \subset s_1 \cup s_2 \cup \dots \cup s_i$  and  $\mathcal{B}_i \subset \bar{s}_i$  for every  $i \in I$ , then there is an  $n$ -cube of  $K$  intersecting all sets  $s_i, \bar{s}_i$ .*

**Proof.** This theorem is a duality-based translation of Theorem 1.7 into the language of  $K$ . Let  $s_{n+1} = \bar{s}_1 \cap \dots \cap \bar{s}_n$  and  $d_i = *s_i$  for every  $i = 1, 2, \dots, n+1$ . Then the sets  $d_1, d_2, \dots, d_{n+1}$  form a covering of  $Q$ . Lemma A.2.1 implies that  $A_i \subset d_1 \cup d_2 \cup \dots \cup d_i$  and  $d_i$  is disjoint from  $B_i$  for every  $i \leq n$ . By Theorem 1.7 this implies that the intersection  $d_1 \cap d_2 \cap \dots \cap d_{n+1}$  is non-empty. It remains to apply Lemma A.2.2. ■

**A duality-based approach to Lebesgue theorems.** In Section 7 we presented a proof of Lebesgue first covering theorem (which immediately implies the second one) based on the multiplication of cochains. A duality-based version of this approach allows to move to  $Q$  a little bit earlier and prove a combinatorial analogue of Theorem 7.8. Here it is.

**A.2.5. Theorem.** *Suppose that  $d_1, d_2, \dots, d_n$  are cubical subsets of  $Q$ . If*

$$A_i \subset d_i \quad \text{and} \quad B_i \subset \bar{d}_i$$

*(where  $\bar{d}_i$  is the union of all  $n$ -cubes of  $Q$  not contained in  $d_i$ ) for every  $i \in I$ , then*

$$(d_1 \cap d_2 \cap \dots \cap d_n) \cap (\bar{d}_1 \cap \bar{d}_2 \cap \dots \cap \bar{d}_n) \neq \emptyset.$$

**Proof.** For each  $i \in I$  let  $c_i \subset K$  be such that  $*c_i = d_i$ . Let  $\bar{c}_i = K \setminus c_i$ . Then

$$*(\bar{c}_i) = \bar{d}_i.$$

Lemma A.2.1 implies that  $\mathcal{A}_i \subset c_i$  and  $\mathcal{B}_i \subset \bar{c}_i$  for every  $i \in I$ . Therefore the sets  $c_i, \bar{c}_i$  satisfy the assumptions of Theorem 7.7. By this theorem there exists an  $n$ -cube intersecting all these sets. It remains to apply Lemma A.2.2. ■

**Deducing Lebesgue theorems from Theorem A.2.5.** First of all, one can deduce Theorem 7.8 from its combinatorial analogue (i.e. from the last theorem) in exactly the same manner as Lebesgue first covering theorem was deduced from its combinatorial analogue (i.e. from Theorem 1.6). We leave details to interested readers. After this one can prove Lebesgue first covering theorem in exactly the same way as in Section 7.

**Duality in arbitrary dimensions.** Suppose that

$$(29) \quad \sigma = \prod_{i=1}^n \rho_i,$$

is a cube of  $K$ . Thus  $i \in I$  either  $\rho_i = \{a_i, a_i + 1\}$  for a non-negative integer  $a_i \leq k - 1$ , or  $\rho_i = \{a_i\}$  for a non-negative integer  $a_i \leq k$ . Let

$$*\rho_i = \{a_i + 1\} \quad \text{if} \quad \rho_i = \{a_i, a_i + 1\},$$

$$*\rho_i = [a_i, a_i + 1] \quad \text{if} \quad \rho_i = \{a_i\},$$

and

$$*\sigma = \prod_{i=1}^n *\rho_i.$$

Clearly,  $*\sigma$  is a cube of  $Q$ , called the *dual cube* of  $\sigma$ . If  $\sigma$  is an  $m$ -cube, then  $*\sigma$  is an  $(n - m)$ -cube.

A cube  $\sigma$  of  $K$  is uniquely determined by its dual cube  $*\sigma$ , and a cube  $c$  of  $Q$  has the form  $*\sigma$  if and only if  $c$  is not contained in  $\text{bd } Q$ . We will also say that the cube  $\sigma$  is the *dual cube* of  $*\sigma$ . If  $c = *\sigma$ , we will also write  $\sigma = *c$ . So,  $\sigma = **\sigma$  if  $\sigma$  is a cube of  $K$ , and  $c = **c$  if  $c$  is a cube of  $Q$  not contained in  $\text{bd } Q$ .

**A.2.6. Lemma.** *Suppose that  $\sigma, \tau$  are two cubes of  $K$ . Then  $\sigma$  is a face of  $\tau$  if and only if  $*\tau$  is a face of  $*\sigma$ .*

**Proof.** Let  $\sigma$  be an  $m$ -cube of  $K$  given by the product (29) and let  $\tau$  be an  $(m - r)$ -face of  $\sigma$  for some  $r \leq m$ . Then  $\tau$  can be obtained by replacing  $r$  two-point sets

$$\rho_i = \{a_i, a_i + 1\}$$

in (29) by either  $\{a_i\}$ , or  $\{a_i + 1\}$ , and  $*\tau$  can be obtained by replacing  $r$  one-point sets

$$*\rho_i = \{a_i + 1\}$$

in the product defining  $*\sigma$  by the intervals

$$[a_i, a_i + 1] \quad \text{or} \quad [a_i + 1, a_i + 2]$$

respectively. Clearly,  $*\sigma$  is a face of  $*\tau$ . A similar argument shows that  $\tau$  is a face of  $\sigma$  if  $*\sigma$  is a face of  $*\tau$ . The lemma follows. ■

**Corollary.** Suppose that  $v_1, v_2, \dots, v_r$  are vertices of an  $m$ -cube  $\sigma$  of  $K$ . Then the  $n$ -cubes  $*v_1, *v_2, \dots, *v_r$  of  $Q$  have a common  $(n - m)$ -face.

**Proof.** Indeed,  $*\sigma$  is a face of  $*v_i$  for every  $i \leq r$ . ■

**Poincaré duality.** Let us indicate some deeper aspects of relations between  $K$  and  $Q$ .

Let us consider the cubes of  $K$  as chains and the cubes of  $Q$  as cochains. A cochain of  $Q$  is called a *relative cochain* of  $(Q, \text{bd } Q)$  if it is equal to a formal sum of cubes of  $Q$  not contained in  $\text{bd } Q$ . As linear functionals  $C_{n-m}(Q) \rightarrow \mathbb{F}_2$ , relative cochains are functionals vanishing on the subspace  $C_{n-m}(\text{bd } Q)$ . The duality map  $\sigma \mapsto *\sigma$  is a bijection between  $m$ -cubes of  $K$  and  $(n - m)$ -cubes of  $Q$  not in  $\text{bd } Q$  and hence leads to an isomorphism

$$C_m(K) \rightarrow C^{n-m}(Q, \text{bd } Q).$$

If a cube of  $Q$  is contained in  $\text{bd } Q$ , then all its faces are contained in  $\text{bd } Q$ . It follows that the coboundary operators  $\partial^*$  define maps

$$\partial^*: C^k(Q, \text{bd } Q) \rightarrow C^{k+1}(Q, \text{bd } Q),$$

also called *coboundary operators*. As it turns out, the duality map turns the coboundary operators  $\partial^*$  into the boundary operators  $\partial$ . In more details, if  $\gamma$  is a chain of  $K$ , then

$$(30) \quad *(\partial(\gamma)) = \partial^*(*\gamma).$$

This immediately follows from Lemma A.2.6.

One can also proceed in other direction and start with the duality map  $c \mapsto *c$  from cubes of  $Q$  to cubes of  $K$ . The only difficulty is the fact that  $*c$  is not defined if  $c$  is a cube of  $Q$  contained in  $\text{bd } Q$ . The solution is to set in this case  $*c = 0$ . Let us introduce the spaces

$$C_m(Q, \text{bd } Q) = C_m(Q) / C_m(\text{bd } Q)$$

of *relative chains* of  $(Q, \text{bd } Q)$ . The duality map  $c \mapsto *c$  leads to an isomorphism

$$C_m(Q, \text{bd } Q) \rightarrow C^{n-m}(K).$$

The boundary operators define maps

$$\partial: C_m(Q, \text{bd } Q) \rightarrow C_{m-1}(Q, \text{bd } Q),$$

also called *boundary operators*. Lemma A.2.6 implies that the identity (30) holds also when  $\gamma$  is a relative chain of  $(Q, \text{bd } Q)$ .

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